Exercise Sheet #1

1. Let Z_G be the center of a group G and assume G/Z_G is cyclic, then show that G is abelian.

2. Let V_1 and V_2 be finite dimensional vector spaces over a field k. Consider the set $Hom_k(V_1, V_2)$ of k-linear maps between V_1 and V_2.

   - Consider the natural right action of $GL(V_1)$ on $Hom_k(V_1, V_2)$ and give a criterion for two map $f, g \in Hom_k(V_1, V_2)$ to be in the same $GL(V_1)$ orbit.
   - If $V_2 = k$, how many $GL(V_1)$ orbits are there?

3. Let S be a set along with a transitive action of a group G. Let H be the stabilizer of a point $s \in S$. Show that the double cosets of H in G are in bijection with orbits of the G action on $S \times S$ given by $g * (s_1, s_2) = (g \cdot s_1, g \cdot s_2)$.
   Using the above show that $\mathrm{GL}_n(\mathbb{R}) = \sqcup_{w \in S_n} BwB$, where B is the subgroup of upper triangular matrices.

4. Let k be a field and V be a vector space of dimension $n > 0$ over F. Assume $|k| > 2$ and let $GL(V)$ denote k-linear isomorphisms of V.

   - Use the fact that $GL(V)$ acts transitively on the set of lines to compute the center of the group $GL(V)$.
   - Consider a decomposition $V = \oplus_{i=1}^{k} V_i$ into subspaces V_i of dimension $e_i > 0$. Let $H \leq GL(V)$ be the subgroup that preserves this decomposition i.e. the H action sends V_i to iteself for all i. Then compute the normalizer $N_{GL(V)}(H)$ and the centralizer $Z_{GL(V)}(H)$ of H in $GL(V)$ and describe their quotient. (It might be useful to first consider the case when $\dim V_i = 1$ for all i.)
   - Compute the normalizer of S_n in $GL_n(\mathbb{R})$. Here S_n is the subgroup of permutation matrices.

5. Let G be the group generated by x and y along with relations x^2 and y^2. Show the following:

   - G is isomorphism to the group generated by reflection s_1 and s_2 about line ℓ_1 and ℓ_2 whose angle of intersection is not a rational multiple of 2π. Conclude that G is infinite.
   - Let N be any proper normal subgroup, then G/N is a dihedral group.

6. Show that any map between a final object F and an initial object I in any category $\mathcal{C}$ is an isomorphism.

7. Let G be a group and consider the category BG which has a single object $*$ and the morphisms $Hom_{BG}(*, *) := G$. Show that giving a G action on objects in a category $\mathcal{C}$ is equivalent to a functor from $BG \to \mathcal{C}$.

8. Let k be a field and G be a finite group and consider the catgory $\text{Rep}(G)$ whose objects are k-vector spaces V along with a G action i.e $G \to Aut_k(V)$. Let Vec_k be the category of vector spaces over k and consider the functors:

$$triv : \text{Vec}_k \to \text{Rep}(G), \quad (V \to (V, G \to \{Id_V\}))$$

$$inv : \text{Rep}(G) \to \text{Vec}_k, \quad ((V, \rho : G \to Aut(V)) \to V^G := \{v \in V | gv = v \ \forall g \in G\})$$

$$coinv : \text{Rep}(G) \to \text{Vec}_k, \quad ((V, \rho : G \to Aut(V)) \to V_G := V/(\text{Span}\{gv - v \ \forall g \in G\}))$$

Show that $(coinv, triv)$ and $(triv, inv)$ are adjoint pairs.

9. Let $T : \mathcal{C} \to \mathcal{C}$ be an endofunctor. A *coalgebra* for an endofunctor T is an object $C \in \mathcal{C}$ equipped with a map $\gamma : C \to T(C)$. A morphism $f : (C, \gamma) \to (C', \gamma')$ of coalgebras is a map $f : C \to C'$ such that $\gamma' \circ f = T(f) \circ \gamma$.

   Show that if (C, γ) is a final object in the category of coalgebras, then the map $\gamma : C \to T(C)$ is an isomorphism.

10. A category C is called skeletal if it only contains just one object for each isomorphism class. The skeleton $sk(\mathcal{C})$ of a category $\mathcal{C}$ is the unique upto isomorphism skeletal caegory that is equivalent to $\mathcal{C}$.

    - Given a category $\mathcal{C}$, construct it's skeleton.
    - Consider the category Fin_{iso} whose objects are finite sets and morphisms are bijections. Construct the skeleton of the categry Fin_iso.

11. Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, then

    - Let f be a morphism in $\mathcal{C}$ such that $F(f)$ is an isomorhism, then f is an isomorphism
    - If A and B are objects in $\mathcal{C}$, so that $F(A) \simeq F(B)$, then A and B are isomorphic in $\mathcal{C}$.
    - Show that the converse holds for any functor.

12. Use the Yoneda theorems to show the following:

    - Any group is isomorphic to a subgroup of the permutation group.
    - Every row operation on matrices with n rows is defined by left multiplication by some n × n matrix, namely the matrix obtained by performing the row operation on the identity matrix.

13. Let (L, R) be a adjoint pair of funtors between $\mathcal{C}$ and $\mathcal{D}$. Assuming limits exists show that for a family of objects $\{A_i\}_{i \in I}$. (Hint: Use Yoneda)

$$R(\varprojlim A_i) = \varprojlim R(A_i)$$