Exercise Sheet #2

1. Show that if $G/Z(G)$ is nilpotent, then G is nilpotent.
2. Show that there is no group whose commutor subgroup is S_4.
3. Let G be a finite group and H be a normal subgroup of G. Let P be a Sylow p-subgroup of H. Then $G = HN_G(P)$ and $|G : H|$ divides $|N_G(P)|$.
4. Let G be a finite group and $\Phi(G)$ be the intersection of all maximal subgroups of G. Show the following:
   (a) $\Phi(G)$ is nilpotent if G is finite.
   (b) $\Phi(G)$ is preserved by any automorphism of G.
   (c) If N is normal in G, show that $\Phi(N) \leq \Phi(G)$.
5. Let G be a finite group in which every proper subgroup is nilpotent, then G is solvable.
6. Let V be a finite dimensional vector space over the field of p-elements and let $\varphi \in GL(V)$ be an element of order p. Prove that there is some non-zero element $v \in V$ such that $\varphi(v) = v$.
7. Any group of order $p^e a$ where $1 \leq a < p$ is not simple. Prove that simple groups of order < 60 are of prime order.
8. Compute $G/Z(G)$ where G is the Dihedral group of order 2^{n+1}.
9. Prove that A_n is the only subgroup of S_n of index 2.
10. Show that the length of the upper and the lower central series of a nilpotent group agree. Give an example to show that the upper and lower central series may not have the same terms
11. Show by induction of length of the upper and lower central series of a nilpotent group G that for any proper subgroup H of G, the normalizer $N_G(H)$ contains H properly.