Exercise Sheet #3

1. (Not for submission)Problem 15 and Problem 17 from Atiyah McDonald Commutative algebra Chapter 1.

2. Let R be a commutative rings and G be a finite group. Then show that the center $Z(R[G])$ is a free R module of finite rank. Give an explicit basis for $G = S_3$.

3. Let R be a ring (not necessarily commutatitve). Show that there is a one-one correspondence between a decomposition of R into a direct sum of left ideal $I_1 \oplus \cdots \oplus I_n$ and elements $e_1, \ldots, e_n \in R$ such that $e_i^2 = e_i$ and $e_i e_j = 0$. (Such an R is called semisimple).

4. Let k be a field whose characteristic is not 2 or 3. Then using the above show that $k[S_3]$ is semisimple.

5. Let R be an integral domain such that every non-zero proper ideal factors uniquly into a product of prime ideal. (Such rings are called Dedekind domains). Then show the following:
   (a) If R is a UFD, then R is also a PID.
   (b) If R is Dedekind if and only if R is Noetherian and localization at any maximal ideal is a local PID.

6. Let R be a domain and let M be am R module. Show that M is torsion free if and only if $M_{\mathfrak{m}}$ is torison free for all maximal ideals $\mathfrak{m}$ in R. Use this to show that if R is a dedekind domain and M is torsion free, then M is flat.

7. (Lang Page 168)Let M be a projective finite module over a Dedekind domain R. Show that there exists free modules F and F' such that $F' \subset M \subset F$, where F and F' have the same rank. Use this to show that there exists a basis $e_1, \ldots, e_n$ and ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ such that $M = \oplus_i \mathfrak{a}_i$.

8. (Read Lang Chapter III, Section 9 and Section 10)
   (a) Do problem 21 in Chapter III about direct limits preserving exactness.
   (b) Problem 22 (a) and (b) in Chapter III about the Hom functor and direct sum and products.

9. (Lang Chapter III Problem 17)Let p be a prime number and consider the ring of p-adic integers $\mathbb{Z}_p$ defined in class. Show the following:
   (a) $\mathbb{Z}_p$ is a local PID with maximal ideal $(p)\mathbb{Z}_p$.
   (b) Show that $\mathbb{Z}_p/(p)\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$.

(c) Consider the inverse system $\{\mathbb{Z}/n\mathbb{Z}\}$ given by map $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ if n divides m. Show that the inverse lim of the inverse system $\varprojlim \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$.

10. Let R be a PID and K be its fraction field. Consider $G \subset GL_n(K)$ whose matrix entries have a common denominator in R. Then show that G is conjugate to a subgroup of $GL_n(R)$. (Hint: Consider $M = \sum_{g \in G} g(R)$ as an R module and show that $\varphi : R^n \simeq M$ is free. Use the isomorphism to construct a matrix $A = (\varphi(e_1), \dots, \varphi(e_n))$).

11. Let V be a finite dimensional vector space over k. If $A \in End_k(V)$ is cyclic, then show that the $BA = AB$ if and only if we can find a polynomial p such that $B = p(A)$.

12. Let V be finite-dimensional over $\mathbb{C}$. Let $A \in End_{\mathbb{C}}(V)$ be such that $V \simeq \mathbb{C}[t]/(t-\lambda)^e$. What is the Jordan form of A^2. Use the answer to decide when a matrix $B \in M_(\mathbb{C})$ has a square root, i.e. when is there A such that $A^2 = B$?

13. Let k be an algebraic closed filed and let V be a finite dimensional K vector space and let A be an endomorphism of V. Use the Jordon decomposition write $V \cong \bigoplus V_i$ where $V_i \cong k[x]/(t-\alpha_i)^{e_i}$. Define A_s to be the endomorphism of V which acts by multiplication by α_i on V_i. Then show the following:

    (a) $A_n := A - A_s$ is nilpotent, i.e. $A^N = 0$. for some $N > 0$.

    (b) $A_s A_n = A_n A_s$

    (c) There exist polynomials p_s and p_n such that $A_s = p_s(A)$ and $A_n = p_n(A)$.

14. Let $SL_n(R) := \{A \in GL_n(R) | \det A = 1\}$. Then show the following:

    (a) If R is an Euclidean domain, then $SL_n(R)$ is generated as a group by elementary matrices.

    (b) Show that the same for any local ring R. (Recall a local ring has a unique maximal ideal. Try to adapt after suitable modification the argument using matrices done in class).

    (c) Use it to show that the reduction $SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/n)$ is surjective.

15. (Lang 29 Chapter III) Let k be a field of characteristic 0. Let $\mathfrak{n}$ be the set of all strictly upper triangular matrices (i.e. diagonal and below entries are zero) of size $n \times n$.

    (a) Let $D_1(X), \dots, D_n(X)$ be the diagonals of entries X in $\mathfrak{n}$. By definition $D_1(X) = 0$. Let $\mathfrak{n}_i$ be the subset of $\mathfrak{n}$ consisting of matrices whose diagonal $D_1, \dots, D_{n-i}$ are zero. Show that $\mathfrak{n}_i$ is an algebra.

    (b) Let U be the set of elements of the form $I + X$ for $X \in \mathfrak{n}$, then show that U is an multiplicative group.

    (c) Show that the exponential map gives a bijection between $\mathfrak{n} \to U$.

    (d) Show that U contains no nontrivial elements of finite order.