Exercise Sheet #4

1. If V is an n dimensional vector space, then show that $(\bigwedge^k V)^* \cong \bigwedge^{n-k} V$ for all $0 \leq k \leq n$.

2. Show the following in $R[[t]]$ for any $A \in M_n(R)$ where R is a commutative ring with identity.

   - $\det(I_n + tA) = \sum_{i=0}^{n} \text{Trace}(\wedge^i A)t^i$
   - $\det(I_n - tA)^{-1} = \sum_{i \geq 0} \text{Trace}(Sym^i A)t^i$

3. Consider the category of Abelian groups and the functor between $Ab^{opp} \to Ab$ given by $A \to Hom_{Ab}(A, \mathbb{Q}/\mathbb{Z})$. Show that this defines and equivalence between $(TorAb)^{opp}$ of the opposite category of torsion abelian groups and the category of profinite groups.

4. Let $\mathcal{A}$ be a abelian category. For $A, B \in Ob(\mathcal{A})$ an extension E of A by B is a short exact squence $0 \to B \to E \to A \to 0$ in $\mathcal{A}$. Two extensions E and E' are equivalent if there exists an extension E'' and map $E'' \to E$ and $E'' \to E'$ such that the following commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \uparrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E'' & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \end{array} \tag{1}$$

   - Show that this is an equivalence relation and let $Ext(A, B)$ denote the set of equivalence class of extensions of A by B.
   - Show that $Ext : \mathcal{A}^{opp} \times \mathcal{A} \to Sets$ is an bifunctor
   - Show that $Ext(A, B)$ is a group structure defines a bifunctor $Ext : \mathcal{A}^{opp} \times \mathcal{A} \to Ab$
   - Show that if $F : \mathcal{A} \to \mathcal{B}$ is fully faithful, then $Ext(A, B) \to Ext(F(A), F(B))$ induced by F is injective.

5. Let V be a finite dimensional vector space over $\mathbb{R}$ and let $\xi \in V$ be a non-zero vector. Consider the map $\xi\wedge : \bigwedge^p V \to \bigwedge^{p+1} V$. Extend this to a map $\xi\wedge : \bigwedge^\bullet V \to \bigwedge^\bullet V$ and let $\iota(\xi)$ denote the transpose of $\xi\wedge$. Show the following:

   - $\iota(\xi)(v_1 \wedge \cdots \wedge v_p) = \sum_{i=1}^{p} (-1)^{i+1} v_1 \wedge \ldots v_{i-1} \wedge \iota(\xi)(v_i) \wedge v_{i+1} \wedge \cdots \wedge v_p$
   - $\iota(\xi)(u \wedge v) = \iota(\xi)(u) \wedge v + (-1)^{deg(u)} u \wedge \iota(\xi)(v)$

   ($\iota(\xi)$ is an example of an antiderivation.)

6. Let V be a finite dimensional real inner product space with the inner product denoted by $\langle \ , \ \rangle$. Extend it to all of $\bigwedge^\bullet V$ by the formula

$$\langle w_1 \wedge \cdots \wedge w_p, v_1 \wedge \dots v_p \rangle = det\langle w_i, v_j \rangle$$

   (a) Find an orthonormal basis of $\bigwedge^\bullet(V)$ interms of a orthonormal basis $e_1, \dots, e_n$ of V.

   (b) Since $\bigwedge^n(V)$ is one dimensional, then $\bigwedge^n V \backslash \{0\}$. An orientation of V is defined to be a choice of a component. Let V be an oriented vector space, then define an operator $* : \bigwedge^\bullet V \to \bigwedge^\bullet V$ which satisfy the following requirement (This is called the star operator):

   - $*(1) = e_1 \wedge \cdots \wedge e_n$ and $*(e_1 \wedge \cdots \wedge e_n) = 1$
   - $*(e_1 \wedge \cdots \wedge e_p) = \pm e_{p+1} \wedge \cdots \wedge e_n$, where $\pm$ is determined by the component $e_1 \wedge \cdots \wedge e_n$ lies.
   - Show that on $\wedge^p V$ we have $** = (-1)^{p(n-p)}$.
   - Show that $\langle v, w \rangle = *(w \wedge *v) = *(v \wedge *w)$

7. Let V be a real inner product space as above let $\xi \in V$ be a non-zero vector. Consider the left exterior multiplication $\xi\wedge : \wedge^p V \to \wedge^{p+1} V$ and let γ denote its adjoint operator. Show that for any $v \in \wedge^p V$

$$\gamma(v) = (-1)^{np} * (\xi \wedge (*v))$$

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9. Defining Grothendieck groups and rings:

   (a) Let A^+ be a commutative monoid. Then show that there exists an abelian group A along with a monoid morphism $\iota : A^+ \to A$ which has the following universal property that for any abelian group B if $f : A^+ \to B$, then f extends to a unique group morphism $\widetilde{f} : A \to B$. The unique upto isomorphism group is called the Grothendieck group.

   (b) If $a + b = a + c$ implies $b = c$ for $a, b, c \in A^+$, then show that ι is injective.

   (c) Repeat (a) for a commutative semi-ring R^+ and construct a ring R with above universal properties.

10. For any commutative ring R consider the monoid $\widehat{W}^+(R)$ of non-degenerate quadratic forms upto isomorphisms and using (a) define $\widehat{W}(R)$ to be the Grothendieck-Witt group of R. The Witt ring $W(R)$ of R is defined to be the quotient of $\widehat{W}(R)$ by the ideal generated by hyperbolic forms. Show the following:

   - If $char(k) \neq 2$, then $\widehat{W}^+(k) \hookrightarrow \widehat{W}(k)$.
   - With the same hypothesis, show that $W(k)$ is spanned classes of ansiotropic forms.
   - Show that two anisotropic forms are isometric if they give same elements in the Witt ring.