Exercise Sheet #5

1. Let R be a domain and let I be a non-zero finitely generated injective R-module, then R is a field.

2. Let R be a domain which is not a field and M be an injective object which is also projective, then $M = 0$.

3. Let $P^\bullet$ be a bounded above complex of projective R-modules. Show that it is null-homotopic iff it is acyclic.

4. Let $\pi : P \to C \to 0$ be a surjection from a projective object P to an object C in an abelian category $\mathcal{A}$ with $K = Ker\pi$. Let A be another object in $\mathcal{A}$ and define

$$_\pi \operatorname{Ext}(C, A) = \operatorname{coker}(\operatorname{Hom}(P, A) \to \operatorname{Hom}(K, A)),$$

Show that there is a group isomorphism between $_\pi Ext(C, A) = Ext(C, A)$ where the last group was as defined in Homework set 4.

5. Compute the following:

   - $Ext(\mathbb{Z}/n\mathbb{Z}, A)$, where A is an abelian group.
   - Let $R = \mathbb{Z}/4\mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z}$, compute $Ext^i_R(M, M)$ and $Tor^R_i(M, M)$

6. (No need to submit)Let R be a commutative rings and E be a free module of finite rank r with a map $s : E \to R$, consider the following (known as Koszul Complex)

$$K_\bullet(s) : \bigwedge^r E \to \bigwedge^{r-1} E \to \cdots \to E \to R \to 0$$

with the differential given by $d_k(s)(e_1 \wedge \cdots \wedge e_k) = \sum_{i=1}^k (-1)^{i+1} s(e_i) e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n$.

   (a) If M is a finitely generated R module, then set $K_\bullet(s, M) := K_\bullet(s) \otimes_R M$with the differential $d(v \otimes m) = d(v) \otimes m$ where d is the usual Koszul differential. Compute the zero-th and r-th homology of $K_\bullet(s, M)$ in terms of R, s and M.

   (b) Let $t \in R$ and consider the Koszul complex for $(s, t) : E \oplus R \to R$ and denote it by $K(s, t)$. Show that there is a short exact sequence of complexes

$$0 \to K_\bullet(s) \to K_\bullet(s, t) \to K_\bullet(s)[-1] \to 0$$

   Give an explicit formula for the connecting homomorphism. You might need a formula for $\wedge^k(V_1 \oplus V_2) = \oplus_{i+j=k} \wedge^i V_1 \otimes \wedge^j V_2$.

(c) Let M be a finitely generated R module. An element of $a \in R$ is said to be non-zero divisor on M if $am = 0$ implies $m = 0$ for $m \in M$. A sequence $x_1, \dots, x_r$ of elements of R is called M-regular if x_i is a non-zero divisor in $M/(x_1, \dots, x_{i-1})M$. Show by induction on r that

$$H_i(K(x_1, \dots, x_r; M)) = 0 \quad \text{for } i > 0$$

Pay special attention to the case $r = 1$.

(d) Let R, M be as above and $x_1, x_2, \cdots, x_n$ a sequence of elements of R. Suppose there is a ring S, an S-regular sequence $y_1, y_2, \cdots, y_n$ in S and a ring homomorphism $S \mapsto R$ that maps y_i to x_i. Then show that

$$H_i(K(x_1, \dots, x_n; M)) = Tor_i^S(S/(y_1, \dots, y_n), M)$$

7. Let A be a commutative $\mathbb{C}$ algebra and $D_1, \dots, D_k$ are mutually commuting operators on A. Consider the complex $\bigwedge_A^{\bullet}(A^k)$ where the differential is given by

$$d(fe_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{i=1}^{k} D_i(f) e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}$$

Show that if any of the D_i's are bijective, then the complex is acyclic.

8. Let R be a commutative ring and M be a free R module of finite rank n. Let $\varphi : G \to Aut_R(M) = GL(n, R)$ be a representaion of G. Consider $R_\epsilon = R[\epsilon]/(\epsilon^2)$. An infinitesimal deformation of φ is a map $\varphi_\epsilon : G \to GL_{n,R_\epsilon}$ such that setting $\epsilon = 0$ recovers φ from φ_ϵ. Two deformations φ'_ϵ and φ''_ϵ are said to be equivlant if there is a $A \in GL_n n, R_\epsilon$ such that $A_{|\epsilon=0} = I_n$ that satisfy $A\varphi'_\epsilon A^{-1} = \varphi''_\epsilon$ for all $g \in G$. Show that equivalence classes are in bijection with $Ext_{RG-mod}(M, M)$.

9. Consider an exact sequence in the category of R-modules

$$0 \to M_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0,$$

where P_i's are all projective modules and M_n is the kernel of $P_{n-1} \to P_{n-2}$. Show that for $i \geq 1$

$$\mathrm{Ext}_R^i(M_n, N) \cong Ext_R^{i+n}(M, N)$$

where Ext^i is defined using projective resolutions of M.

10. (No need to submit) Consider an exact sequence in the category of R-modules

$$0 \to N \to I^0 \to \cdots \to I^{n-1} \to N_n \to 0,$$

where I^i's are all projective modules and N_n is the kernel of $I^{n-2} \to I^{n-1}$. Show that for $i \geq 1$

$$\mathrm{Ext}_R^i(M, N_n) \cong Ext_R^{i+n}(M, N)$$

where Ext^i is defined using injective resolutions of N.