Exercise Sheet #6

1. Let X be a left R-module and consider the Tor functor $T(X)(B) := B \otimes_R X$ from the category of right R modules to the category of abelian groups and define $\operatorname{Tor}_i^R(X, B) := L_i T(X)(B)$ and can be computed using a projective resolution of B. Let S be a ring which is a projective R-module. (In the Tor notation keep the left module to the left and right module to the right)

   Now show the following:

   (a) Let M be a right R-module and N be a left S-module. Show that
   $$\operatorname{Tor}_i^R(N, M) \cong \operatorname{Tor}_i^S(N, M \otimes_R S)$$
   for all i.

   (b) Let M be a left S-module and N a left R-module. Show that
   $$\operatorname{Ext}^i_R(M, N) \cong \operatorname{Ext}^i_S(M, {}_R\operatorname{Hom}(S, N)).$$

   Hint: As done in class, use the result about computing derived functor of composition of a exact and a half exact functor.

2. Let G be a group and let M be a left $\mathbb{Z}G$-module. Define $H^i(G, M) = \operatorname{Ext}^i_{\mathbb{Z}(G)}(\mathbb{Z}, M)$.

   (a) Show that $H^0(G, M) = M^G$.

   (b) Show that for any abelian group A, the G-module $\operatorname{Hom}(\mathbb{Z}G, A)$ is acyclic. Use it to construct a resolution for any G-module acyclic with respect to the functor of taking G-invariants.

3. Let G be a group and let M be a right $\mathbb{Z}G$-module.

   (a) Define $H_i(G, M) = \operatorname{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, M)$ and show that the coinvariants M_G is given by $\operatorname{Tor}_0^{\mathbb{Z}G}(\mathbb{Z}; M) = M \otimes_{\mathbb{Z}G} \mathbb{Z}$.

   (b) Show that for any subgroup H of G and any right $\mathbb{Z}H$ module M, we get $\operatorname{Tor}_i^{\mathbb{Z}H}(\mathbb{Z}, M) \cong \operatorname{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, M \otimes_{\mathbb{Z}H} \mathbb{Z}G)$.

   (c) For any abelian group A, show that $A \otimes_{\mathbb{Z}} \mathbb{Z}G$ is acyclic for the functor of taking G-coinvariants.

4. Prove that for a PID, torsion freeness of a module is equivalent to flatness. (outlined in class)

5. Let S be a multiplicative set in a ring R, show that for all $n \geq 0$ and all R modules A and B we have isomorphisms
   $$S^{-1} \operatorname{Tor}_n^R(A, B) \cong \operatorname{Tor}_n^{S^{-1}R}(S^{-1}A, S^{-1}B)$$
   Use this to conclude that if $A_\mathfrak{m}$ is a flat $R_\mathfrak{m}$ module for every maximal ideal $\mathfrak{m}$, then A is flat.

6. (Kunneth for cohomology: No need to submit but do it) Let R be a PID and assume that $X_\bullet$ is a complex of free left R modules and $Y^\bullet$ is any cochain complex. Define a new complex $\operatorname{Hom}(X_\bullet, Y^\bullet)^n := \prod_i \operatorname{Hom}(X_i, Y^{n-i})$ along with the obvious differential

   Show that there is a short exact sequence
   $$0 \to \prod_{p-q=n-1} \operatorname{Ext}^1(H_p(X_\bullet), H^q(Y^\bullet)) \to H^n(\operatorname{Hom}(X_\bullet, Y^\bullet)) \to \prod_{p-q=n} \operatorname{Ext}^0(H_p(X_\bullet), H^q(Y^\bullet)) \to 0$$
   which splits non-canonically. For the next two problems, you will need the above (see Rotman Chapter 10, Page 680-685)

7. Show that if R is a PID, and A, B and C are R-modules, then there is an isomorphism
   $$\operatorname{Ext}^1_R(\operatorname{Tor}_1^R(A, B), C) \cong \operatorname{Ext}^1_R(A, \operatorname{Ext}^1_R(B, C))$$

8. (No need to submit but do it)With the same assumptions as before, show that
   $$\operatorname{Ext}^1(A \otimes_R B, C) \oplus \operatorname{Hom}_R(\operatorname{Tor}_1^R(A, B), C) \cong \operatorname{Ext}^1_R(A, \operatorname{Hom}_R(B, C)) \oplus \operatorname{Hom}_R(A, \operatorname{Ext}^1_R(B, C))$$