Exercise Sheet #7

1. If a and b are elements in a ring R. Show that $1 - ba$ is left invertible implies $1 - ab$ is left invertible.
2. Show that for a prime p, the group $G = \{\frac{a}{p^n} | a \in \mathbb{Z},\ n \in \mathbb{Z}_{\geq 0}\}/\mathbb{Z}$ is an Artinian $\mathbb{Z}$-module.
3. Let R be a ring, show that the categories of R modules is equivalent to the categroy of $\mathrm{Mat}_n(R)$-modules. (Morita Equivalence).
4. Let p be a prime and Let G be a p group of cardinality p^r for $r \geq 1$. Let k be a field of characteristic p. Compute $k[G]/J(k[G])$.
5. (Double Centralizer Property)Let R be a simple ring and $I \subset R$ a non-zero left ideal. Then $R \to \mathrm{End}_{R'^{opp}}(I)$ is an isomorphism, where $R' = \mathrm{End}_R(I)$. (Hint: The steps discussed in class. This is a generalization of a well known fact about rings of matrices)
6. Use above to show that if R is simple, then TFAE
   (a) R has a minimal nonzero left ideal
   (b) R is left Artinian
   (c) R is semisimple
   (d) $R \cong M_n(\mathbb{D})$, for some $n \geq 1$ and some division algebra $\mathbb{D}$.
7. Let R be a ring and I be an ideal consisting of nilpotent elements. Show that
   (a) If $\bar{e}$ is an idempotent in R/I, then it can be lifted to an idempotent $e \in R$.
   (b) If in addition R is commutative, then $R \to R/I$ is a bijection on idempotents.
8. Show that $J(M_n(R)) = M_n(J(R))$. If I is a minimal left ideal, then is $M_n(I)$ minimal in $M_n(R)$?
9. Find the Artin-Wedderburn decompostion of $\mathbb{C}[D_{2.4}]$
10. Let $\Gamma = \mathbb{Z}/p\mathbb{Z}$ and embed Γ as subgroup pf $GL_2(\mathbb{F}_p)$ as upper triangular matrices with 1 along the diagonals. Let $M = \mathbb{F}_p^2$ and consider the action of Γ on M induced by the left multiplication by $GL_2(\mathbb{F}_p)$. Find a Jordan-Holder series for M. Is M semisimple as $\mathbb{F}_p\Gamma$ module.
11. Consider the $\mathbb{R}$ algebra $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ with multiplication with the multplication of quaternions. Show the following:
    - $\mathbb{H}$ is a division algebra and $H \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$ as $\mathbb{C}$-algebras
    - Let Q be the subgroup $\pm i, \pm i, \pm j, \pm k$ of $\mathbb{H}^*$ and let $R = R[Q]$ and $R_{\mathbb{C}} = \mathbb{C}[Q]$. Compute the following:
      (a) Write all simple R-modules upto isomorphism
      (b) Write $R_{\mathbb{C}}$ as a product of simple $\mathbb{C}$-algebras.
12. (Not for submission) Let M be an R module and define $Rad(M) := \cap\{N \subset M | N \text{ is maximal}\}$.
13. (Not for submission) Show that for every M module R we have $Rad(Soc(M)) = 0$. Show that if M is Artinian, then $Rad(M) = 0$ if and only if it is semisimple.
14. (Not for submission) (Socle Series)Let M be an R-module and define $Soc_1(M) = Soc(M)$. Define $Soc_{n+1}(M)$ recursively by the formula $Soc_{n+1}(M)/Soc_n(M) \cong Soc(M/Soc_n(M))$. Show the following:
    (a) (Not for submission) Compute Socle of $\mathbb{Z}$.
    (b) (Not for submission) If R is Artinian, then show that $Soc_i(M) = \{m \in M | J^i m = 0\}$.
    (c) (Not for submission) If M is Artinian show that the Socle series stabilizes iff there exists n such that $Soc_n(M) = M$.

(d) (Not for submission) Let R be a ring and M be an left R module of finite length. Consider a finite length filtration

$$0 \subset M_1 \subseteq \cdots \subset M_i \subseteq \cdots \subseteq M$$

such that the corresponding quotients are simple. Show that M_i is not contained in $Soc_{i-1}M$.

(e) (Not for submission) Work out the previous part (d) under the assumption R is Artinian.

15. (Frobenius Reciprocity) Recall if A, B, C are three rings and let ${}_AM_B$ be an (A, B) bimodule, ${}_BN_C$ be an (B, C) bimodule and ${}_AK_C$ be a (A, C)-bimodule. Then there are isomorphisms (proved in class, so use it directly)

$$\mathrm{Hom}_C(M \otimes_B N, K) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_C(N, K))$$

$$\mathrm{Hom}_A(M \otimes_B N, K) \cong \mathrm{Hom}_B(N, \mathrm{Hom}_A(M, K))$$

(a) Use the above to show that $(Ind_H^G, Res_H^G, Coind_H^G)$ form an adjoint triple. i.e.

$$Hom_G(Ind_H^G(N), M) = Hom_H(N, Res_H^G(M))$$

$$Hom_G(Res_H^G(M), N) = Hom_H(M, Coind_H^G(N))$$

where M and N are left H and G modules respectively.

(b) Let M and N be left H and G modules respectively, then show that

$$Ind_H^G(M \otimes Res_H^G(N)) = Ind_H^G(M) \otimes N$$