Exercise Sheet #8

1. Let $G = \langle a, b | a^5 = b^4 = 1, bab^{-1} = a^2 \rangle$. Compute the character table of G.

2. Compute the character table of $\mathrm{SL}_2(\mathbb{F}_3)$. (Hint: In both these problems try to express it as semidirect products)

3. Let V be an irreducible complex representation of a finite group G with character χ. Let e_V be the primitive idempotent corresponding to V in $\mathbb{C}[G]$. Show that
$$e_V = \frac{1}{|G|} \sum_{g \in G} \chi(1)\chi(g^{-1})g \in \mathbb{C}[G].$$

4. Let G be a finite group and let I be a subset of the set of characters corresponding to irreducible complex representations. Consider the subset of G
$$N_I = \{g \in G | \chi_i(g) = \chi_i(1) \ \forall \chi_i \in I\}.$$
Show that N_I is normal and every normal subgroup is of the above form. Deduce that G is simple iff $\ker \chi_i = \{1\}$ for all irreducible characters χ_i.

5. Let ρ be a complex representation of a finite group G and χ be its character. Define the center of ρ as
$$Z(\rho) := \{g \in G | \rho(g) = \lambda I\}$$
Then show the following:
   (a) $Z(\rho)/Ker\chi$ is cyclic and is contained in $G/Ker\chi$.
   (b) If ρ is irreducible, the above inclusion is an isomorphism.
   (c) The center of G is $\cap_{\rho \in Irr(G)} Z(\rho)$.

6. (not for submission)Let (ρ, V) be a complex self dual representation of a finite group G. Then show the following:
   (a) The characters are real
   (b) Show that there exist a real vector G-stable space W of dimension V such that $V = W \oplus \sqrt{-1}W$ if and only if V admits a non-degenerate symmetric G-invariant bilinear form.

7. If V is a finite dimensional complex representation of a group G, show the following
   (a) $\langle \chi_{V \otimes V}, \mathbf{1} \rangle_G = 1$ if $V \cong V^*$ and is zero otherwise
   (b) The following decomposition exists as $k[G]$-modules:
$$V \otimes V = Sym^2 V \oplus \wedge^2 V$$
   (c) Show that $\chi_{Sym^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$ and $\chi_{\wedge^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$

8. (Not for submission)Consider the Frobenius-Schur Indicator of a complex valued representation χ of a finite group G defined by
$$s_2(\chi) := \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$
   (a) Compute s_2 is χ is not real valued.
   (b) Compute s_2 is V is self dual but V is not realizable over $\mathbb{R}$.
   (c) Compute s_2 if V is self dual and V is realizable over $\mathbb{R}$.
   (d) Compute s_2 for $\mathrm{SL}_2(\mathbb{F}_3)$ and its irreducible representations.
   (e) Let $r : G \to \mathbb{N}$ be the function
$$r(g) = \#\{h \in G | h^2 = g\} \in \mathbb{N}.$$
   Prove that $r(g) = \sum_{\chi \in Irr(G)} s_2(\chi)\chi(g)$.