Home Work 1

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogenous function i.e. $f(tx) = t^s f(x)$ for all $t > 0$. Show that $\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = sf$.

2. Consider the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ on $\mathbb{R}^3$. Express the Laplacian in the spherical polar coordinates.

3. Check that the spherical Laplacian $-\Delta_{S^2}$ is self adjoint and positive semidefinite with respect to the Hermitian form.

4. Let $\mathcal{H}_k$ denote the space harmonic polynomials in $\mathbb{C}[x, y, z]$ of degree k. Show that SO(3) acts on $\mathcal{H}_k$.

5. Consider the special unitary group SU(2) and consider $t_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$. Show that every element of SU(2) is conjugate to an element of the form t_ϕ.

6. Consider the natural action of $\mathbb{C}^2$. Show the following;
   (a) SU(2) acts on the space P_k of homogenous polynomials of degree k in $\mathbb{C}[x, y]$.
   (b) Compute the action of t_ϕ on $x^k y^{n-k}$ for $0 \le k \le n$.
   (c) Show that the representation $\pi_k : \mathrm{SU}(2) \to \mathrm{GL}_n(P_k)$ is irreducible.

7. Consider the Lie algebra $\mathfrak{su}(2)$ of SU(2) and construction an isomorphism of $\mathfrak{su}(2)$ with the Lie algbera $\mathfrak{so}(3)$ of $\mathrm{SO}(3, \mathbb{R})$. (Hint: Construct a basis of both the Lie algebras and compute their Lie brackets.)

8. Show that the conjugation $X \to AXA^{-1}$ defines an action of the group SU(2) on the vector space $\mathfrak{su}(2)$. Further show the following:
   (a) The conjugation action gives a map $\phi : \mathrm{SU}(2) \to \mathrm{GL}_3(\mathbb{R})$.
   (b) Show that the image of ϕ is a subgroup of SO(3).
   (c) Using Problem 7 and the exponential map, show that the map ϕ is surjects onto SO(3)
   (d) Show that the Kernel is $\mathbb{Z}/2$.

9. Using Problem 7 show that π_{2k} factorizes to give an irreducible representation of SO(3).