Home Work 10

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1. Let L_i be the dual of the diagonal matrix H_{ii} in $\mathfrak{gl}_n(\mathbb{C})$.Give a description of the root lattice and the weight lattice of $\mathfrak{sl}_n(\mathbb{C})$ in terms of L_i's. Also express the fundamental weights of $\mathfrak{sl}_n(\mathbb{C})$ in terms of the L_i's. What is index of the root lattice in the weight lattice?

2. Repeat the above exercise for $\mathfrak{sp}(2n)$ and $\mathfrak{so}(2n+1)$. (The case for roots was discussed in class).

3. Show that for a n-dimensional vector space V, the exterior powers $\wedge^i V$ are fundamental representation of $\mathfrak{sl}(V)$.

4. Let $\lambda = \sum_{i=1}^{n-1} a_i\omega_i$ be a dominant integral weight of $\mathfrak{sl}(V)$. Show that the irreducible highest weight moduli V_λ is a submodule of
$$\mathrm{Sym}^{a_1}(\wedge^1 V) \otimes \dots \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1} V)$$

5. Show that for any i,j such that $i+j \leq n$, the $\mathfrak{sl}(V)$ module V_λ is in the Kernel of all natural maps of $\mathfrak{sl}(V)$ modules
$$\mathrm{Sym}^{a_1}(\wedge^1 V) \otimes \dots \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1} V) \to$$
$$\wedge^i V \otimes \wedge^j V \otimes \mathrm{Sym}^{a_1}(\wedge^1 V) \otimes \cdots \otimes \mathrm{Sym}^{(a_i-1)}(\wedge^i V) \otimes \cdots \otimes \mathrm{Sym}^{a_j-1} \wedge^j V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1} V)$$

6. Assume $\dim V = 4$. Describe the weights of the representations $\mathrm{Sym}^n V$ and $\mathrm{Sym}^n(\wedge^2 V)$ and deduce that they are irreducible and reducible respectively.

7. Let V be a $2n$ dimensional vector space with a non-degenerate symplectic form, show that the representation of $\mathfrak{sp}(V)$ on V is irreducible.

8. Show that the representation of $\mathfrak{sp}(V)$ on the vector space $\wedge^2 V$ is not irreducible.

9. For any $1 \leq k \leq n$ consider the map $\phi_k : \wedge^k V \to \wedge^{k-2} V$ defined by
$$\phi_k(v_1 \wedge \cdots \wedge v_k) = \sum_{i<j} Q(v_i, v_j)(-1)^{i+j-1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k$$
Show that the k-th fundamental representation is contained in the Kernel of ϕ_k. (It turns out that the k-th fundamental represention is the whole Kernel. (See Fulton-Harris or argue with Weyl dimension formula.)