Home Work 2

The first five problems are from the book of Brocker–tom-Dieck.

1. (Gram-Schmidt process) Let D be the subgroup of $\mathrm{SL}(n,\mathbb{R})$ consisting upper triangular matrices with positive entries on the diagonal. Show that the multiplication map $D \times \mathrm{O}(n) \to \mathrm{GL}(n,\mathbb{R})$ is a diffeomorphism.
2. Show that the exponential map is surjective for $\mathrm{SO}(n,\mathbb{R})$ and $\mathrm{U}(n)$.
3. Let G be a Lie group and H be a Lie subgroup of G, then show that H is closed.
4. Let G be a Lie group and G^0 be the connected component at identity. Show that G^0 is a normal Lie subgroup of G.
5. Show that a discrete normal subgroup of a connected Lie group must be contained in the center.
6. Show that the Lie algebra on $T_e\,\mathrm{GL}_n(\mathbb{R})$ is same as the Lie algebra structure on $\mathfrak{gl}(n,\mathbb{R})$.
7. Let $\mathfrak{X}(M)$ be the set of vector fields on a manifold M. Show that the Jacobi identity holds for $\mathfrak{X}(M)$ with respect with the natural bilinear operation.
8. Let $\mathfrak{g}$ be a Lie algebra. For a fixed $X \in \mathfrak{g}$, consider the map $ad_X : \mathfrak{g} \to \mathfrak{g}$ given by $ad_X(Y) := [X,Y]$. Show that
   - ad_X is a derivation of $\mathfrak{g}$.
   - $[D, ad_X] = ad_{D(X)}$ for any derivation $D \in \mathrm{Der}(\mathfrak{g})$.
9. Let A be the ring of C^∞ functions on $\mathbb{R}^{n+n}$ with coordinates $p_1,\ldots,p_n,q_1,\ldots,q_n]$. Define the Poisson bracket on A by the formula for any f and g in A:

   $$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}$$

   Show the following:

   (a) $\{,\}$ is a Lie algebra.

   (b) For $f \in A$, the map $\{f,\} : A \to A$ is a derivation and hence defines a vector field. (Hint: Leibnitz rule for $\{,\}$).
10. (V. Kac)Let A be the ring C^∞ functions on $\mathbb{R}^n$ and fix some assignments $\{x_i,x_j\} \in A$ for every pairs (i,j). Now define the bracket

    $$\{f,g\} = \sum_{i=1,j=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\{x_i,x_j\}$$

    Show that this bracket satisfies the axioms of a Lie algebra if and only if $\{x_i,x_i\} = 0$, $\{x_i,x_j\} = -\{x_j,x_i\}$ and for any triple x_i,x_j,x_k the bracket $\{,\}$ satisfy the Jacobi identity (Hint: Bull headed approach works).