Home Work 3

1. If X is a vector field such that $X_p \neq$ for a point $p \in M$, then there exists a coordinate chart U around p such that $X_{|U} = \frac{\partial}{\partial x_1}$, where x_1 is the first coordinate of U.
2. $X_1, \ldots, X_k$ are local vector field in a neighbourhood of a point p in M such that $[X_i, X_k] = 0$ and $X_1, \ldots, X_k$ are linear indepdent. Show that there exists a coordinate chart U around p such that $X_{i|U} = \frac{\partial}{\partial x_i}$.
3. Let G be a connected Lie group. Show that the center $Z(G)$ is the Kernel of the adjoint representation. Describe the Lie algebra of $Z(G)$ in terms of the Lie algebra of G.
4. Show that the fundamental group of a connected Lie group is abelian. (Hint: Use the exercise about discrete kernel from previous homework).
5. Prove that every matrix in $\mathrm{G}_n(\mathbb{C})$ can be uniquely written as a product PU, where P is a positive define matrix and $U(n)$ is a unitary matrix and use it to show that $\mathrm{GL}_n(\mathbb{C})$ is connected. (Hint: Is $U(n)$ connected?)
6. Is the exponential map $\mathfrak{su}(2) \to \mathrm{SU}(2)$ open ? Also given an example of a Lie group where the exponential map is not surjective.
7. Let X and Y be two vector field on M and ϕ_t^X and ϕ_t^Y are their flows. Let $p \in M$ be a point and consider the curve
$$\beta(t) = \Phi^Y_{-\sqrt{t}} \Phi^X_{-\sqrt{t}} \Phi^Y_{\sqrt{t}} \Phi^X_{\sqrt{t}}(m).$$
Show that the following limit
$$\lim_{t \to 0} \frac{f(\beta(t) - f(\beta(0)))}{t}$$
exists for $f \in C^\infty(M)$ and equals $[X, Y]_p f$