Home Work 5

The main goal of this problem set is to give a proof of the Campbell-Baker-Hausdorff formula.

Let X be a set and consider the free associative algebra Ass_X and it's completetion F_X. We can think of F_X as the formal power series indexed by X. Let $\mathfrak{m}$ denote the ideal in F_X generated by the image of X. Finally let L_X be the free Lie algebra on a set X and since $L_X = A_X/I$ is graded, we consider its completion and denote it by $\mathbf{L}_X$. We have the following maps

$$\exp : \mathfrak{m} \to 1 + \mathfrak{m} \quad \log : 1 + \mathfrak{m} \to \mathfrak{m}$$

Since we are dealing with formal power series, they are well defined and mutally inverses of each other.

1. Show that exp is a bijection of the set $\alpha \in \mathfrak{m}$ satisfying $\Delta\alpha = \alpha \otimes 1 + 1 \otimes \alpha$ wit the set of all $\beta \in 1 + \mathfrak{m}$ such that $\Delta(\beta) = \beta \otimes \beta$.

2. Show that the set $\beta \in 1 + \mathfrak{m}$ such that $\Delta(\beta) = \beta \otimes \beta$ forms a group. Use it to show that given A, B in $\mathbf{L}_X$, we can find $C \in \mathbf{L}_X$ such that $\exp C = \exp A \exp B$.

3. Define the map $\Phi : \mathfrak{m} \cap Ass_X \to L_X$ by the formula

   $$\Phi(x_1 \dots x_n) := ad(x_1) \dots ad(x_{n-1})(x_n)$$

   and let $\Theta : Ass_X \to \mathrm{End}(L_X)$ be the natural homomorphism that extends the adjoint representation $L_X \to \mathrm{End}(L_X)$. Show that

   $$\Phi(uv) = \Theta(u)\Phi(v), \text{ for all } u \in Ass_X \text{ and } v \in \mathfrak{m} \cap Ass_X$$

4. Let L_X^n denote the n-th graded component of the Lie algebra L_X induced from the graded ideal I, then

   $$\Phi(u) = nu, \quad \text{for } u \in L_X^n$$

   (Hint: Use induction and the Lie bracket and part 3.)

5. Let $x \neq y$ in X and consider $z = \log(\exp x \exp y) = \sum_{n=1}^{\infty} z_n(x, y)$, where z_n is a sum of non-commutative monomials of degree n in the variables x and y. By applying part 4 show that $z_n = \frac{1}{n} \sum_{p+q=n} z'_{p.q} + z''_{p,q}$, where

   $$z'_{p,q} := \sum_{\substack{p=p_1+\dots+p_m; q-1=q_1+\dots+q_{m-1} \\ p_i+q_i \geq 1, p_m \geq 1, q_m = 1}} \frac{(-1)^{m+1}}{m} \frac{(ad(x))^{p_1}(ad(y))^{q_1} \dots ad(x))^{p_{m-1}}(ad(y))^{q_{m-1}}(ad(x))^{p_m} y}{p_1!q_1! \dots p_{m-1}!q_{m-1}!p_m!} \tag{1}$$

   $$z'_{p,q} := \sum_{\substack{p-1=p_1+\dots+p_{m-1}; q=q_1+\dots+q_{m-1} \\ p_i+q_i \geq 1, p_m = 1, q_m = 0}} \frac{(-1)^{m+1}}{m} \frac{(ad(x))^{p_1}(ad(y))^{q_1} \dots ad(x))^{p_{m-1}}(ad(y))^{q_{m-1}} x}{p_1!q_1! \dots p_{m-1}!q_{m-1}!p_m!} \tag{2}$$

   (Hint: Expand $\log(\exp x \exp y)$ and first investigate why after applying part 4 to the degree n-components why $q_m \geq 2$ can't appear and similarly if $q_m = 0$ why $p_m \geq 2$ can't appear)

6. Write z_n explicitly for $1 \leq n \leq 5$.