Home Work 6

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1. Consider the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ and the complex Lie group $\mathrm{SL}_2(\mathbb{C})$
   (a) Let P be the set of set of trace zero Hermitian matrices, show that there is an isomorphism of $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{su}(2) \oplus \sqrt{-1}P$.
   (b) Show that there is a isomorphism betwee $\mathrm{SU}(2) \times P \simeq \mathrm{SL}(2,\mathbb{C})$ as real manifolds. (Hint: Similar question was discussed in a earlier homework)
   (c) Show that $\mathrm{SL}(2,C)$ is simply connected.

2. Let G be any complex Lie group with Lie algebra $\mathfrak{g}$. By taking derivatives and restrictions, construct vertical and horizontal arrows in the diagram below such that there are bijective.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}}(\mathrm{SL}_2(\mathbb{C}), G) & \longrightarrow & \mathrm{Hom}_{\mathbb{R}}(\mathrm{SU}_2, G) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{C}}(\mathfrak{sl}_2(\mathbb{C}), \mathfrak{g}) & \longrightarrow & \mathrm{Hom}_{\mathbb{R}}(\mathfrak{su}(2), \mathfrak{g}) \end{array}$$

3. Use the above to show that the finite dimensional representations of SL_2, $\mathfrak{sl}(2)$, $\mathfrak{su}(2)$ and $\mathrm{SU}(2)$ are in bijective correspondence.

4. Let $V(m) := \mathrm{Sym}^m \mathbb{C}^2$ be the representation of $\mathrm{sl}(2)$. For any two non-negative integers $mlgeqn$ show that
$$V(m) \otimes V(n) \simeq V(m+n) \oplus V(m+n-2) \cdots + \oplus V(m-n)$$
as $\mathfrak{sl}(2)$-modules. (Hint: In class, we did $m = n = 1$, but don't try to work with a basis of $(\mathbb{C}^2)^{\otimes m}$. Instead use the basis given in the notation of highest weight vectors)