# Home Work 7

The exercises are all from Humphreys's book on Lie algebras.

1. Let $u \in \mathrm{End}(V)$, where $\dim V < \infty$ and $u = s + n$ be its decomposition into semi-simple and nilpotent parts. Show that $ad_u = ad_s + ad_n$ is the decomposition of $ad_u$ considered as operators in $\mathrm{End}(\mathrm{End}(V))$.
2. Let $\mathfrak{U}$ is a finite dimensional algebra over a field $k$. Show that $\mathrm{Der}(\mathfrak{U}) \subset \mathrm{End}(\mathfrak{U})$ contains the semi-simple and nilpotent parts of all its elements
3. Using Cartan's criterion, show that a Lie algebra $\mathfrak{g}$ such that $\mathrm{Tr}(ad_x ad_y) = 0$ for all $x, y \in \mathfrak{g}$ is solvable.
4. Let $k$ be a field of prime characteristic $p > 0$. Consider the $p \times p$ matrix $A$ such that $A_{p,1} = 1$ and $A_{i,i+1} = 1$ for $1 \leq i \leq p-1$ and rest of the entries 0 and $B = \mathrm{diag}(0, 1, 2, \ldots, p-1)$. Check that $A$ and $B$ span a two dimensional solvable Lie subalgebra $\mathfrak{gl}(p, k)$. Verify that $A$ and $B$ have no common eigen vector.
5. Let $k$ be a field of characteristic $p \neq 0$. Show that $\mathfrak{g}$ is semi-simplie if its Killing form is non-degenerate.

   Let $\mathfrak{g} = \mathfrak{sl}(3, k)$ and $char(k) = 3$. Show that the converse fail for $\mathfrak{g}/Z(\mathfrak{g})$
6. Show that $(\rho, V)$ is a representation of a semisimple Lie algebra $\mathfrak{g}$, then $\rho(\mathfrak{g}) \subset \mathfrak{sl}(V)$. In particular any one dimensional $\mathfrak{g}$-module has trivial action.
7. Let $\beta(x, y)$ and $\alpha(x, y)$ be two symmetric associative bilinear forms on a simple Lie algebra $\mathfrak{g}$. If $\beta$ and $\alpha$ are non-degenerate, then $\beta$ and $\alpha$ are proportional.

   For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, then show that $\kappa(x, y) = 2n\,\mathrm{Tr}(xy)$, for all $x, y \in \mathfrak{g}$ where $\kappa$ is the Cartan-Killing form.
8. Define a Lie algebra $\mathfrak{g}$ to be reductive if $Z(\mathfrak{g}) = \mathrm{rad}(\mathfrak{g})$. Show that a representation $V$ of $\mathfrak{g}$ is completely reducible if $Z(\mathfrak{g})$ acts on $V$ by semi-simple endomorphisms.
9. (Preservation of Jordan Decomposition, Humphreys Section 6.4) It was proved in class that derivations of a semi-simple Lie algebra $\mathrm{Der}(\mathfrak{g}) = \mathfrak{g} \subset \mathrm{End}(\mathfrak{g})$ via the adjoint action.

   Show using Problem 2, that for any $x \in \mathfrak{g}$, there exists $n, s \in \mathfrak{g}$ such that $ad_x = ad_s + ad_n$ is the decomposition into semi-simple and nilpotent parts in $\mathrm{End}(\mathfrak{g})$.
10. Define an abstract Jordan decomposition of $x = s + n$ where $x, s, n$ as in Porblem 9. However if $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ for some $V$, this presents an ambiguity. We will now show that there is no ambiguity in the next steps.
    (a) Let $\mathfrak{g} \subset \mathrm{End}(V)$ be a semisimple Lie algebra and consider $x \in \mathfrak{g}$ and consider the usual Jordan decomposition $x = s + n$. Then show that $s, n \in N_{\mathfrak{gl}(V)}(\mathfrak{g})$, where $N$ denotes the normalizer.
    (b) Show that $\mathfrak{g} \subsetneq N_{\mathfrak{gl}(V)}(\mathfrak{g})$.
    (c) Let $W$ be a $\mathfrak{g}$ submodule of $V$ and define
    $$\mathfrak{g}_W := \{y \in \mathfrak{gl}(V) | y(W) \subset W \text{ and } Tr(y|W) = 0\}.$$
    Show that $\mathfrak{g} \subset \mathfrak{g}_W$.
    (d) Define $\mathfrak{g}' := N_{\mathfrak{gl}(V)}(\mathfrak{g}) \bigcap (\cap_W \mathfrak{g}_W)$, where the intersection is taken over all irreducible subrepresentations of $W$ of $V$. Show that $s, n \in \mathfrak{g}'$.
    (e) Now using complete reduciblity show that $\mathfrak{g}' = \mathfrak{g}$.
    (f) Use this to show that the abstract and the usual Jordan decomposition coincide.
11. Let $\phi : \mathfrak{g} \to \mathrm{End}(V)$ be a representation of a semi-simple Lie algebra $\mathfrak{g}$. If $x = s + n$ is the abstract Jordan decomposition, then show that $\phi(x) = \phi(s) + \phi(n)$ is the usual Jordan decomposition of $\phi(x)$.