Home Work 8

---

1. Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ be the subset consisting of matrices of the form

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

where A is a $k \times k$ matrix, B is a $k \times (n-k)$ matrix and D is a $(n-k) \times (n-k)$ matrix.

   (a) Show that $\mathfrak{g}$ is a Lie algebra (algebras over this form are called parabolic subalgebras).

   (b) Compute the radical $rad(\mathfrak{g})$ of $\mathfrak{g}$ and describe $\mathfrak{g}/rad(\mathfrak{g})$.

2. Let $\mathfrak{g}$ be a real Lie algebra with a positive definite bilinear form, then $\mathfrak{g} = 0$.

3. Let W be a subrepresentation of a representation V for a Lie algebra $\mathfrak{g}$, then show that the bilinear

$$(\ , \ )_V = (\ , \ )_W + (\ , \ )_{V/W}$$

4. Show that the complexification $\mathfrak{sl}(2, \mathbb{C})$ considered as a real Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$.

5. Let $\mathfrak{g}$ be a Lie algebra and V be a an irreducible $\mathfrak{g}$-module. Let K be the ring of $\mathfrak{g}$-endomorphism of V. Show that K is a field.

6. Let $\mathfrak{g}$ be a semi-simple Lie algebra and let K be the ring of the $\mathfrak{g}$-endomorphisms of $\mathfrak{g}$ (with adjoint representations). Let $\overline{k}$ be an algebraic closure of k.

   (a) Let $k = \overline{k}$ and let $\mathfrak{g} = \oplus_{i=1}^{h} \mathfrak{s}_i$, where $\mathfrak{s}_i$ are simple. Show that K is isomorphic to h copies of product of k.

   (b) (No assumptions on k) Let h be the number of components of $\mathfrak{g} \otimes \overline{k}$, then $[K : k] = h$ and K is a product of m-commutative fields, where m is the number of simple components of $\mathfrak{g}$.

   (c) A Lie algebra $\mathfrak{g}$ is called absolutely simple if $\mathfrak{g} \otimes \overline{k}$ is simple. Show that this is equivalent to $K = k$.