Let R be a finitely generated k-algebra.

Recall. $\mathrm{Der}(R) = \{ \theta : R \to \mathbb{C} \mid \theta$ is $\mathbb{C}$-linear
$\theta(fg) = \theta(f)g + \theta(g)f \}$.

$R = k[x_1, \dots, x_n]$.

Consider. $D_R = \bigoplus_{\alpha \in \mathbb{N}^n} k[x_1, \dots, x_n] \, \partial_{\vec{x}}^{\alpha}$. $\partial_x^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$
$\alpha \in \mathbb{N}^n$.

D_R acts on R

$P \in D_R$ and let $P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial_x^\alpha$. we define the total symbol. $\sigma(P)(x, \xi) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \xi^\alpha$.

where. $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$.

$P_1, Q_1 \in D_R$, then $\sigma(P_1 Q_1)$?

$$\sigma(P_1 Q_1)(x, \xi) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma(P)(x, \xi) \partial_x^\alpha \sigma(Q).$$

$\alpha! = \alpha_1! \dots \alpha_n!$

There is a natural filtration.

$$F_\ell D_R = \sum_{|\alpha| \le \ell} R . \partial_x^\alpha \quad \left(\ell \in \mathbb{N} \text{ and } |\alpha| = \sum \alpha_i \right)$$

<u>Properties</u> a) $D_R = \bigcup_{\ell \in \mathbb{N}} F_\ell D_R$.

b). $F_\ell . D_R$ is ~~both~~ free module.

c). $F_0 D_R = R$, $(F_\ell D_R) \times (F_m D_R) = F_{\ell+m} D_R$

d) $P \in D_R \in F_\ell(D_R)$ and $Q \in F_m D_R$
$[P, Q] \in F_{\ell+m-1} D_R$

$$\mathrm{gr}\, D_R = \mathrm{gr}^F D_R = \bigoplus_{\ell=0}^{\infty} \mathrm{gr}_\ell D_R . \qquad \mathrm{gr}_\ell .\, D_X = F_\ell D_R / F_{\ell-1} D_R .$$

$F_{-1} D_R = 0.$

<u>Def</u> $\xi_i = \partial_i \bmod F_0 D_R \in \mathrm{gr}_1 D_R .$

<u>Then</u> $$\mathrm{gr}_\ell D_R = \frac{F_\ell D_R}{F_{\ell-1} D_R} \simeq \bigoplus_{|\alpha| = \ell .} \mathcal{O}_U \xi^\alpha$$

<u>Remark</u>: $\mathrm{gr}\, D_R$ is a finitely generated comutative algebra. over R. — $R[\xi_1, \dots, \xi_n]$. =

So. what are ξ_i — <u>tangent vector field</u>

<u>Remark</u>: X be a manifold., then, let D_X be the ring of differential operator on a manifold. $\mathrm{gr}\, D_X = \mathrm{Sym}^\bullet T_X$ (Symmetric algebra).

- (<u>Poincare – Birkhoff Witt Theorem</u>) for arbitary Lie algebra.
- We will replace D_X by an object known as universal <u>enveloping algebra</u>