Free Lie algebra

Magma $M \times M \to M$. (map. $(x,y) \to xy$).

X be any set. $X_1 := X$.

$$X_n := \bigsqcup_{p+q=n} X_p \times X_q$$

X_2 consist of all expressions (a,b)

$$M_X = \bigsqcup_{n\geq 1}^{\infty} X_n.$$

$X_3 = \{ ((a,b),c)$
$(a,(b,c))$

$X_4 = \{ ((a,b),c)\,d$
$(a\,(b,c))\,d.$
$(a,b)\,(c,d)$
$(a,(b,c,d))\}$
$(a\,(b,c))\,d.$

$$X_p \times X_q \hookrightarrow X_{p+q}.$$

This M_X has a multiplication.
just contatination of words.

If N is any magma, $f : X \to N$ is any map

$F : M_X \to N$ as a map of magmas.

$F : X_2 \to N \quad F(ab) = f(a)\,f(b)$

$F : X_p \times X_q \quad F(u,v) = F(u)\,F(v).$

A_X be formal linear combinations of elements of $M_X \Rightarrow A_X$ is an algebra.

Similarly

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & B \\ M_X \arrow[ur, "F"] \arrow[r] & A_X \arrow[u, "\tilde{F}"'] \end{tikzcd}$$

B is any algebra.

$\tilde{F}: A_X \to B$ is the unique algebra map.

Since every element of M_X has a length

$\Rightarrow$ A_X is graded

Free Lie algebra X be a set.

I be the two sided ideal generated by

a) aa $a \in X$

◦) $(ab)c + (bc)a + (ca)b$. $a, b, c \in A_X$.

$$L_X = A_X / I$$ is called the free Lie algebra

Observe: Any map from $X \to \mathfrak{g}$ extends to a Lie algebra homomorphism from $L_X \to \mathfrak{g}$,

Thm: $$L_X = \{ w \in Ass(X) \mid \Delta(w) = w \otimes 1 + 1 \otimes w \}$$

$$\mathcal{U} L_X \simeq Ass(X) = T(V_X).$$

V_X is the free vector space on X.

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A_X is graded and I is the ideal formed by elements of the form aa $a \in A_X$ and $(ab)c + (bc)(a) + (ca b)$ $a, b, c \in A_X$.

<u>Claim</u> : I is graded.

$a = \sum a_n$. into homogeneous components.

$a_n \in I$

$J \subset I$ denote the set $a = \sum a_n$. with $a_n \in I$.

Clearly J is a twosided ideal in A_X and $J \subset I$.

We need to show $J \supseteq I$.

If $a = \sum a_p$, $b = \sum b_q$, $c = \sum c_r$.

$(ab)c + (bc)a + (ca)b$

$$= \sum_{p,q,r} (a_p b_q) c_r + (b_q c_r) a_p + (c_r a_p) b_q.$$

If

$x = \sum x_m$.

$$x^2 = \sum x_n^2 + \sum x_m x_n + x_n x_m.$$

$$x_m x_n + x_n x_m = (x_m + x_n)^2 - x_m^2 - x_n^2 \in I$$

so $I \subset J$

$L_X = A_X / I$ is a graded Lie algebra.

where $Ass_X = T(V_X)$.

$$\begin{tikzcd} L_X \arrow[r, hook, "i"] & \mathcal{U}(L_X) \arrow[d, "\Phi"] \\ X \arrow[u] \arrow[ur] \arrow[r] & Ass_X \arrow[u, bend right, "\Psi"'] \end{tikzcd}$$

$\Phi \circ \Psi$ and $\Psi \circ \Phi$ are identity on X.

$\Rightarrow \quad \underline{Ass_X \simeq \mathcal{U}(L_X)}$

<u>Questn</u>

$L_X \subset \mathcal{U}(L_X)$, inj $\Rightarrow$ $L_X \hookrightarrow Ass_X$ is also injective

<u>Claim</u>: $L_X = \{ w \in Ass_X \mid \Delta(w) = w \otimes 1 + 1 \otimes w \}$.

$$\Delta : X \longrightarrow Ass_X \otimes Ass_X$$

$$x \longrightarrow x \otimes 1 + 1 \otimes x.$$

$$\Delta : Ass_X \longrightarrow Ass_X \otimes Ass_X$$

$$\Delta : \mathcal{U}(L_X) \longrightarrow \mathcal{U}(L_X) \otimes \mathcal{U}(L_X)$$

Hence the result follows.

An element is called primitive if

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

Claim: $\mathfrak{g}$ are all the primitive in $U\mathfrak{g}$.

Case $\mathfrak{g}$ is abelian $\mathcal{U}(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g})$. Then $\mathrm{Sym}(\mathfrak{g}) \otimes \mathrm{Sym}(\mathfrak{g})$ as polynomial in twice the number of variables as that of $\mathrm{Sym}(\mathfrak{g})$ and

$$\Delta f(u,v) = f(u+v)$$

Primitive

$$f(u+v) = f(u) + f(v).$$

Taking homogeneous components and taking $u=v$.

$$\Rightarrow f(2u) = 2f(u).$$

$\Rightarrow$ f is linear.

$\mathfrak{g}$ is not abelian, taking gr and applying PBW.

$\Rightarrow$ the primitive are contained in $\mathcal{U}_1\mathfrak{g}$.

But $\Delta(c+x) = c(1\otimes 1) + x\otimes 1 + 1\otimes x.$

So the condition on primitive require $c = 2c$.

$\Rightarrow c = 0$ char $\neq 2$.