$$D^0 \mathfrak{g} = \mathfrak{g}, \; D^1 \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}], \; \ldots, \; D^{n+1} \mathfrak{g} = [D^n \mathfrak{g}, D^n \mathfrak{g}]$$

Exercise: $\mathfrak{b}$ be the Lie algebra of upper triangular matrices

$$D^+ \mathfrak{b} = \mathfrak{n}_+.$$

Def: A Lie algebra $\mathfrak{g}$ is solvable if either of the following holds

(1) $\exists n$ such that $D^n \mathfrak{g} = 0$.

(2) $\exists n$ such that every family of 2^n elements of $\mathfrak{g}$, the successive brackets of bracket vanish.

(3) $\mathfrak{g} \supset I_1 \supset I_2 \supset \cdots \supset I_n = 0$ such that each I_j is an ideal in the preceding such that the quotient I_k / I_{k+1} is abelian ie $[I_k, I_l] \subset I_{k+l}$.

Thm (Lie) Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field k of characteristic zero. (ρ, V) be a finite dimensional representation of $\mathfrak{g}$. Then we can find a basis of V, such that $\rho(g)$ consist of upper triangular matrices

Thm: Under the same hypothesis, there exists a vector (non zero) which is a common eigen vector for all $\rho(y)$ where y is an element of $\mathfrak{g}$. ie there is a vector $v \in V$ and a function $\chi : \mathfrak{g} \to k$, such that

$$\rho(y) v = \chi(y) v \ \forall y \in \mathfrak{g}. \quad (*)$$

Lemma Suppose $I \subset \mathfrak{g}$ be an ideal such that $(*)$ holds for all $y \in I$, then

$$\chi([x, h]) = 0 \ \forall \ x \in \mathfrak{g}, \ h \in I.$$

Proof For $x \in \mathfrak{g}$ let V_i be the subspace spanned by $v, xv, \ldots, x^{i-1} v$ and let $n > 0$ be the minimal V_i such that $V_n = V_{n+1}$. So V_n is finite dimensional and $x V_n \subset V_n$. Also $V_n = V_{n+k} \ \forall k$.

For $h \in I$

$$hv = \chi(h) v.$$

$$hxv = xhv - [x, h] v$$

$$= \chi(h) xv \bmod V_1$$

$$hx^2 v = \chi(h) x^2 v \bmod V_2$$

$$\vdots$$

$$hx^i v = \chi(h) x^i v \bmod V_i.$$

Thus V_n is invariant under I and for each $h \in I$,

$$tr|_{V_n} h = n\chi(h).$$ In particular both x and h leave V_n invariant and $tr|_{V_n} [x,h] = 0$ since trace of any commutator is zero.

This proves the lemma.

<u>Proof of theorem:</u> By induction on $\dim \mathfrak{g}$, which we assume to be positive, let $\mathfrak{m}$ be any subspace of $\mathfrak{g}$ with $\mathfrak{g} \supset \mathfrak{m} \supset [\mathfrak{g},\mathfrak{g}]$. Then $[\mathfrak{g},\mathfrak{m}] \subset [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{m}$
So $\mathfrak{m}$ is an ideal in $\mathfrak{g}$.

In particular, we may choose $\mathfrak{m}$ to a subspace of codimension 1 containing $[\mathfrak{g},\mathfrak{g}]$. By induction we can find $v \in V$ and $\chi : \mathfrak{m} \to \mathbb{R}$ such that

$$S(y)v = \chi(y)v \quad \forall\, y \in \mathfrak{m}.$$

Let $$W := \{ w \in V \mid hw = \chi(h)w, \ \forall\, h \in \mathfrak{m} \}$$

If $x \in \mathfrak{g}$, then

$$hxw = xhw - [x,h]w = \chi(h)xw - \chi([x,h])w$$
$$= \chi(h)\,xw.$$

Since $\chi([x,h]) = 0$, by the Lemma. Thus W is stable under all of $\mathfrak{g}$. Pick $x \in \mathfrak{g} \setminus \mathfrak{m}$ and $v \in W$ be an eigen vector of x with eigenvalue λ.

Then v is a simultaneous eigenvector for all of $\mathfrak{g}$ with χ defined as

$$\chi(h + \gamma x) = \chi(h) + \gamma\lambda.$$

Corollary Let $V = \mathfrak{g}$, and consider the adjoint representation. Then Lie's theorem says that there is a flag of ideals with commutative quotients and hence $[\mathfrak{g},\mathfrak{g}]$ is nilpotent