Complete Reducibility

Prop: Let $\mathfrak{g}$ be a simple Lie algebra and (ρ, V) be a non-trivial representation of $\mathfrak{g}$. Then there exists a central element $C_V \in Z(U\mathfrak{g})$ which acts by a non-zero constant in V.

Pf

$$(x, y)_\rho = tr(\rho(x)\rho(y)) \text{ is an invariant bilinear form.}$$

If $(\,,\,)_\rho$ is non-degenerate, then C_V be the Casimir operator of $\mathfrak{g}$ induced by the form $(\,,\,)_\rho$

If V is a trivial representation, then C_V acts by zero in the trivial representation as all elements of $\mathfrak{g}$ acts by zero.

Now By Schur's Lemma C_V acts as a constant in the irreducible representation V, say $\rho(C_V) = \lambda \, id_V$.

On the other hand.

$$tr(\rho(C_V)) = \sum tr(\rho(x_i)\rho(x^i))$$

$$= \dim \mathfrak{g}.$$

Because $tr(\rho(x_i)\rho(x^i)) = (x_i, x^i)_\rho = 1$

Thus $\lambda = \dfrac{\dim \mathfrak{g}}{\dim V} \neq 0.$

Now if $(,)_{\mathfrak{g}}$ is degenerate. ②

$$I = \ker.(,)_{\mathfrak{g}} \subset \mathfrak{g}.$$

Then it is an ideal in $\mathfrak{g}$ and $I \neq \mathfrak{g}$ (otherwise $\rho(\mathfrak{g}) \subset \mathfrak{gl}(V)$ is solvable).

This is impossible, as it is a quotient of semi-simple Lie algebra and thus itself semi-simple.

By. nondegeneracy of bilinear form, we get

$$\mathfrak{g} = I \oplus \mathfrak{g}'$$

Now $\mathfrak{g}'$ is semi-simple and $B_{V}|_{\mathfrak{g}'}$ is non degenerate.

Now I and $\mathfrak{g}'$ commute; C_V will be central in $U\mathfrak{g}$. and the same argument show that it acts by.

$$\frac{\dim \mathfrak{g}}{\dim V} \neq 0.$$

Theorem Every finite dimensional representation of a semi-simple Lie algebra is completely reducible.

We will use functor $\mathrm{Ext}^i(W,V)$ from homological algebra.

Let V, W are finite dimensional $\mathcal{U}\mathfrak{g}$-modules, we have sequence of abelian groups $\mathrm{Ext}^i(W,V) = \mathrm{Ext}^i_{\mathcal{U}\mathfrak{g}}(W,V)$ $i=0,1,2\ldots$

$\mathrm{Ext}^0(W,V) = \mathrm{Hom}(W,V)$ as

Extensions upto isomorphism

$0 \to V \to E \to W \to 0$. $\mathrm{Ext}^1(W,V)$ are corresponds to.

If $\mathrm{Ext}^1(W,V)=0$, then every extension splits

$$E \cong V \oplus W.$$

Hence we need to prove $\mathrm{Ext}^1(W,V)=0$. $\forall V, W$.

Step I Lemma If V is irreducible then $\mathrm{Ext}^1(\mathbb{C},V)=0$.

Suppose $V=\mathbb{C}$ is trivial.

a) E is a two dimensional representation $\rho(x)$ is strictly uppertriangular for $x \in \mathfrak{g}$.

b) This implies $\rho(\mathfrak{g})$ is nilpotent.

c) Since $\mathfrak{g}$ is semi-simple, $\rho(\mathfrak{g})=0$ d) $E \simeq \mathbb{C}\oplus\mathbb{C}$.

$$0 \to V \to E \to \mathbb{C} \to 0.$$

Suppose V is non trivial.

a) $C_V \in \mathcal{U}g$ that acts as zero in $\mathbb{C}$ and as a nonzero constant λ in V.

b) The Eigen Values of C_V are λ with multiplicity $\dim V$ and 0 with multiplicity 1.

c) $E = V \oplus W$ where W is the 0 eigenspace of C_V.

d) Since C_V is central, W is a subrepresentation.

$$\rho(C_V)e = 0 \Rightarrow \rho(C_V)\rho(x)e$$
$$= \rho(x)\rho(C_V)e$$
$$= 0.$$

e) $P : E \to \mathbb{C}$ takes W isomorphically to $\mathbb{C}$

Thus $E \cong V \oplus \mathbb{C}$

Step 2 Lemma $\mathrm{Ext}^1(\mathbb{C}, V) = 0$ for any representation V (5)

$$0 \to V_1 \to V \to V_2 \to 0.$$

Gives a long exact sequence of Ext groups

$$\to \mathrm{Ext}^1(\mathbb{C}, V_1) \to \mathrm{Ext}^1(\mathbb{C}, V) \to \mathrm{Ext}^1(\mathbb{C}, V_2) \to \cdots$$

$$\Rightarrow \mathrm{Ext}^1(\mathbb{C}, V) = 0. \quad \text{If } \left.\begin{array}{l} \mathrm{Ext}^1(\mathbb{C}, V_1) = 0 \\ \mathrm{Ext}^1(\mathbb{C}, V_2) = 0. \end{array}\right\}$$

This we can argue by induction

Step 3 Lemma $\mathrm{Ext}^1(W, V) = 0$. for any two representations V, W.

Consider an arbitrary extension

$$0 \to V \to E \to W \to 0.$$

Apply the functor $\mathrm{Hom}_{\mathbb{C}}(W, -)$, the produces another short exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{C}}(W, V) \to \mathrm{Hom}_{\mathbb{C}}(W, E) \to \mathrm{Hom}_{\mathbb{C}}(W, W) \to 0.$$

Recall that

$$\mathrm{Hom}(\mathbb{C}, \mathrm{Hom}_{\mathbb{C}}(A, B)) = (\mathrm{Hom}_{\mathbb{C}}(A, B))^{g}.$$

$$= \mathrm{Hom}(A, B).$$

where $(\ ,\ )^{\mathfrak{g}} = \mathrm{Hom}(\mathbb{C}, -)$ is the functor of $\mathfrak{g}$-invariants

(6)

(3) If we apply this to

$$0 \to \mathrm{Hom}_{\mathbb{C}}(W, V) \to \mathrm{Hom}_{\mathbb{C}}(W, E) \to \mathrm{Hom}_{\mathbb{C}}(W, W) \to 0.$$

and recall that $\mathrm{Ext}^1(\mathbb{C}, \mathrm{Hom}_{\mathbb{C}}(W, V)) = 0$,

we see that

$$0 \to \mathrm{Hom}(W, V) \to \mathrm{Hom}(W, E) \to \mathrm{Hom}(W, W) \to 0$$

is still exact.

(4) The fact that

$$0 \to \mathrm{Hom}(W, V) \to \underset{s}{\mathrm{Hom}(W, E)} \to \underset{p \circ s}{\mathrm{Hom}(W, W)} \to 0.$$

is exact that there is a morphism

$s : W \to E$ whose composition with

$p : E \to W$ is id_W.

(5) This gives the splitting $E \simeq V \oplus W$.

Thus we are done.

Application

Suppose $\mathfrak{g}$ is reductive $\Rightarrow$ $\mathfrak{g}/Z(\mathfrak{g})$ is semi-simple

Consider the adjoint representation of $\mathfrak{g}$ on itself.

This makes $\mathfrak{g}$ into a representation of $ad(\mathfrak{g}) \simeq \mathfrak{g}/Z(\mathfrak{g})$

By complete reducibility for $\mathfrak{g}/Z(\mathfrak{g})$ - representations,

we set

$$\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}'$$

where $\mathfrak{g}'$ is a subrepresentation for $ad(\mathfrak{g})$ hence it is an ideal in $\mathfrak{g}$.

This implies $\mathfrak{g}' \longrightarrow \mathfrak{g}/Z(\mathfrak{g})$ is an isomorphism

Therefore $\mathfrak{g}'$ is a semisimple Lie algebra.

$\Rightarrow$ $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}'$, where $\mathfrak{g}'$ is semisimple