Recall the notion of complexification

$$\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_0 \oplus \sqrt{-1}\, \mathfrak{g}_0.$$

Let us list down all real Lie algebra whose complexification is a classical simple Lie algebra. i.e. $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$, $\mathfrak{sp}(2n)$

Real	Complex
$\mathfrak{sl}_n \mathbb{R}$	$\mathfrak{sl}_n \mathbb{C}$
$\mathfrak{sl}_n \mathbb{C}$	$\mathfrak{sl}_n \mathbb{C} \times \mathfrak{sl}_n \mathbb{C}$
$\mathfrak{sl}_n \mathbb{H} = \mathfrak{gl}_n \mathbb{H} / \mathbb{R}$.	$\mathfrak{sl}_{2n} \mathbb{C}$
$\mathfrak{so}_{p,q} \mathbb{R}$	$\mathfrak{so}_{p+q} \mathbb{C}$
$\mathfrak{so}_n \mathbb{C}$	$\mathfrak{so}_n \mathbb{C} \times \mathfrak{so}_n \mathbb{C}$
$\mathfrak{sp}_{2n} \mathbb{R}$	$\mathfrak{sp}_{2n} \mathbb{C}$
$\mathfrak{sp}_{2n} \mathbb{C}$	$\mathfrak{sp}_{2n} \mathbb{C} \times \mathfrak{sp}_{2n} \mathbb{C}$
$\mathfrak{su}_{p,q}$.	$\mathfrak{sl}_{p+q} \mathbb{C}$
$\mathfrak{u}_{p,q} \mathbb{H}$.	$\mathfrak{sp}_{2(p+q)} \mathbb{C}$
$\mathfrak{u}_n^* \mathbb{H}$.	$\mathfrak{so}_{2n} \mathbb{C}$

Thm: (Cartan) This list is complete for $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sp}(2n, \mathbb{C})$.

There are 17 additional simple real Lie algebras associated to ~~some~~ the remaining Lie algebra that show up in Classification

$\mathfrak{so}_{p,q}(\mathbb{R})$ is the Lie algebra of $SO_{p,q}(\mathbb{R})$.

$$SO_{p,q}(\mathbb{R}) = \{ A \in GL(\mathbb{R}) \mid (Av, Aw) = (v, w) \}$$

where $(\ ,\ )$ has p + eigen values
q − eigen values

Remark: $SO_{p,q}(\mathbb{R})$ may not be connected

$\mathbb{H} = \mathbb{C} \oplus j\,\mathbb{C}$ $\quad GL_n\mathbb{H}$ is the quaternionic linear automorphism

$\mathbb{H}$ is 4-dim Real space $\quad GL_n\mathbb{H} \subsetneq GL_{4n}\mathbb{R}$.

Observe $\mathbb{H}$ is not commutative $\to$ $GL_n\mathbb{H}$ we mean it form $\mathbb{H}$-linear automorphism for right $\mathbb{H}$-modules V which is free of rank n

$$\mathbb{H}^n = \mathbb{C}^{2n} = \mathbb{C}^n \oplus j\,\mathbb{C}^n.$$

$\varphi : \mathbb{H}^n \to \mathbb{H}^n$ is $\mathbb{H}$ linear exactly when it commutes with j ie

$$\varphi(vj) = \varphi(v)j \quad \text{If } v = v_1 + jv_2$$

$$v \cdot j = -\bar{v}_2 + j\bar{v}_1$$

So multiplication by j takes

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to \underbrace{\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}}_{J} \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}$$

$$GL_n\mathbb{H} = \{ A \in GL_{2n}(\mathbb{C}) \mid AJ = J\bar{A} \}.$$

$\mathbb{H}$ is the Hamiltonian quaternions.

$$\mathfrak{gl}_n \mathbb{H} = \{ A \in \mathfrak{gl}_{2n}\mathbb{C} \mid AJ = J\bar{A} \} \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$M \in \mathfrak{gl}_{2n}\mathbb{C}$, we can write.

$$M = \tfrac{1}{2}(M - J\bar{M}J) - i\left(\tfrac{1}{2}(iM + iJ \cdot \overline{iM} \cdot J)\right)$$

$$\Rightarrow \mathfrak{gl}_n \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{gl}_{2n}\mathbb{C}$$

Now observe that even for $\mathfrak{sl}(2,\mathbb{C})$, both $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2,\mathbb{C})$ are real forms, where the Lie groups $SU(2)$ $SL(2,\mathbb{R})$ are compact and non compact respectively.

Complexification of Lie groups.

If G is a Lie group, a universal complexification is given by a complex Lie group $G_{\mathbb{C}}$ and a continuous homomorphism $\mathcal{J}: G \to G_{\mathbb{C}}$ such that $f: G \to H$ is an arbitrary continuous homomorphism into a complex Lie group H, then there is a unique complex analytic homomorphism

$$F: G_{\mathbb{C}} \to H \text{ such that}$$

$$f = F \circ \mathcal{J}.$$

Existence : If G is connected with Lie algebra $\mathfrak{g}$.

Consider the universal cover $\tilde{G}$ with Lie algebra $\mathfrak{g}$. Let $\tilde{G}_{\mathbb{C}}$ be the simply connected complex Lie group with Lie algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and $\tilde{\Phi} : \tilde{G} \to \tilde{G}_{\mathbb{C}}$ be the natural map. Lift the map of Lie algebra.

If $G = \tilde{G}$, then $\tilde{G}_{\mathbb{C}}$ is the complexification.

Now if $G \neq \tilde{G}$, and

$f : G \to H$, then this gives a map (unique).

$E : \tilde{G}_{\mathbb{C}} \to H$ such that

$f \circ \pi = E \circ \Phi$ where $\pi : \tilde{G} \to G$.

K = intersection of the kernels of the homomorphisms E and f vary over all possibilities

Then K is a closed normal complex Lie subgroup of $\tilde{G}_{\mathbb{C}}$ and the quotient is the universal complexification

If G is not connected

$$\{1\} \to G^{\circ} \to G \to \Gamma \to \{1\}.$$

this induce an extension $\{1\} \to (G^{\circ})_{\mathbb{C}} \to G_{\mathbb{C}} \to \Gamma \to \{1\}$

Remark : If the original group is linear, then so is the universal complexification and the homomorphism is injective but false otherwise even at Lie algebra level

Examples 1) $G(\mathbb{C}) = SL(2,\mathbb{C})$.

$$\sigma_s(g) = \bar{g}$$

$$G(\mathbb{C})^{\sigma_s} = SL(2,\mathbb{R}).$$

2) $\sigma_c(g) = {}^t\bar{g}^{-1}$, then $G(\mathbb{C})^{\sigma_c} = SU(2)$. (compact)

Def: A compact real form is σ_c on $G(\mathbb{C})$ is an antiholomorphic map such that $G(\mathbb{R}) = G(\mathbb{C})^{\sigma_c}$ is compact real.

Thm: (Cartan) Compact real form exists and is unique upto conjugation by $G(\mathbb{C})$.

Now θ is a holomorphic involution of G, then some conjugate of θ commutes with σ_c. Then after replacing by such the conjugate def $\sigma = \theta \circ \sigma_c$. Then σ is anti holomorphic involution of G. In particular

$$\{\theta \mid \text{holomorphic } \theta^2 = 1\}/G \leftrightarrow \{\sigma \mid \text{anti-holomorphic}, \sigma^2 = 1\}/G.$$

Def: A real form is G-conjugacy class of holomorphic involutions