We know that if A is an unitary matrix, then A is diagonalizable (1)
by conjugation by unitary matrices.

Lecture 27

$T \subset U(n)$ be the subgroup of all diagonal unitary matrices.
It is easy to see that T is a maximal abelian subgroup.
If A is diagonal with distinct eigen values, then any matrix that commutes with A must be diagonal.

In particular any matrix that commutes with all diagonal unitary matrices must itself be diagonal.

As a topological group T is just a product of circles

Let G be a compact Lie group and let T and T' be two maximal tori (So T and T' are commutative subgroup connected and each is not strictly contained in another connected commutative subgroup).

Then $\exists\ a \in G$ such that $a G a^{-1} = T$.

To prove this choose a 1-parameter subgroups of T and T' which are dense in each.

That is chose x and x' in the Lie $G = \mathfrak{g}$.

such that the curve $t \mapsto \exp tx$ is dense in T and $t \mapsto \exp tx'$ is dense in T'

If we could find $a \in G$ such that

$$a(\exp tx')a^{-1} = \exp t\, Ad_a x' \text{ commute with}$$

all the $\exp. sx$, then ~~exp~~ $a(\exp tx')a^{-1}$ will commute with all element of T and hence belong to T. and by continuity

$$aT'a^{-1} \subset T \quad \text{and hence} \quad aT'a^{-1} = T \text{ by maximality}$$

So we would find $a \in G$ such that

$$[Ad_a x', x] = 0.$$

Choose $a \in G$ such that $(Ad_a x', x)$ is a maximum.

Let $y := Ad_a x'$ — and $(\ ,\ )$ is a +ve definite ad invariant form

Claim $[y, x] = 0$.

For any $z \in \mathfrak{g}$ we have

$$([z, y], x) = \frac{d}{dt}\left(Ad_{\exp tz}\, y, x\right)\Big|_{t=0}$$
$$= 0.$$

by the maximality

But $([z, y], x) = (z, [y, x])$

Hence $[y, x]$ is orthogonal to all of $\mathfrak{g}$ and hence 0.

We will give analogies of this statement for Lie algebras over
complex numbers (We work over $\mathbb{C}$ now)

Recall If δ be a derivation of $\mathfrak{g}$.

$\mathfrak{g}_a = \mathfrak{g}_a(\delta)$ be the generalized Eigen space with eigen value a.

- Then $[\mathfrak{g}_a, \mathfrak{g}_b] \subset \mathfrak{g}_{a+b}$ for two eigen values a and b

Let $s(\delta)$ be the semisimple part, hence $s(\delta) = a$. on $\mathfrak{g}_a$.

If $y \in \mathfrak{g}_a$, $z \in \mathfrak{g}_b$.

$$s(\delta)[y, z] = (a+b)[y, z].$$

$$= [s(\delta)y, z] + [y, s(\delta)z].$$

$\Rightarrow$ $s(\delta)$ is a derivation (HW problem).

$\Rightarrow$ $n(\delta)$ is also a derivation.

The nilpotent part and semisimple are derivation.

If $\mathfrak{g}$ is semi-simple $\mathfrak{g} = \operatorname{Der} \mathfrak{g}$.

So any $x \in \mathfrak{g}$ can be written $x = s + n$.

where $s \in \mathfrak{g}$, $n \in \mathfrak{g}$. where $\operatorname{ad} s$ is semisimple
and $\operatorname{ad} n$ is semisimple

Let k be a subalgebra containing $\mathfrak{g}_0(\mathrm{ad}\,x)$, for some $x \in \mathfrak{g}$. (4)

<u>Back to general $\mathfrak{g}$</u>

k be a subalgebra containing $\mathfrak{g}_0(\mathrm{ad}\,x)$ for some $x \in \mathfrak{g}$.

Then $x \in \mathfrak{g}_0(\mathrm{ad}\,x)$ and hence $x \in k$, and hence

$\mathrm{ad}\,x$ preserves $N_{\mathfrak{g}}(k)$. (By Jacobi).

We have

$$x \in \mathfrak{g}_0(\mathrm{ad}\,x) \subset k \subset N_{\mathfrak{g}}(k) \subset \mathfrak{g}.$$

all of them are invariant under $\mathrm{ad}\,x$.

Therefore the ~~zero~~ characteristic polynomial of ad_x restricted to $N_{\mathfrak{g}}(k)$ is a factor of the characteristic polynomial of $\mathrm{ad}\,x$ acting on $\mathfrak{g}$.

But all zero of characteristic polynomial are generated by ~~eigen~~ values of the generalized eigen spaces of $\mathfrak{g}_0(\mathrm{ad}\,x)$.

Hence $N_{\mathfrak{g}}(k)/k$ is acted upon by $\mathrm{ad}\,x$ by nonzero eigen value.

On the other hand $\mathrm{ad}\,x$ acts trivially on the quotient space

If $x \in k$ and hence $[N_{\mathfrak{g}}k, x] \subset k$ by definition.

Hence $\underline{N_{\mathfrak{g}}(k) = k}$. $(**)$

Cartan Subalgebra

A Cartan subalgebra is defined to be a nilpotent subalgebra which is its own normalizer. A Borel subalgebra is defined to be a maximal solvable algebra

Thm: Any two Cartan / Borel subalgebras are conjugate.

Conjugate $\Rightarrow$ $N(\mathfrak{g}) = \{ x \mid \exists\, y \in \mathfrak{g},\ a \neq 0,\ \text{with } x \in \mathfrak{g}_a(\operatorname{ad} y) \}$

Notice every element of $N(\mathfrak{g})$ is ad-nilpotent.

and $N(\mathfrak{g})$ is stable by $\operatorname{Aut}(\mathfrak{g})$

As any $x \in N(\mathfrak{g})$ is nilpotent $\exp \operatorname{ad} x$ is well defined as an automorphism of $\mathfrak{g}$ and we let $\mathcal{E}(\mathfrak{g})$ be the group of automorphisms generated by these element.

Conjugacy means that we can find $\phi \in \mathcal{E}(\mathfrak{g})$ with $\phi(h_1) = h_2$ where h_1 and h_2 are Cartan subalgebra's (Similarly for Borel)

Prop: $\mathfrak{h}$ is CSA iff $\mathfrak{h} = \mathfrak{g}_0(\text{ad } z)$ where $\mathfrak{g}_0(\text{ad } z)$ contains no proper subalgebra of the form $\mathfrak{g}_0(\text{ad } x)$

Proof: $\mathfrak{h} = \mathfrak{g}_0(\text{ad } z)$ which is minimal, then we know that $\mathfrak{h}$ is its own normalizer, (**)

Also By Lemma 1.

$$\mathfrak{h} \subset \mathfrak{g}_0(\text{ad } x) \quad \forall\, x \in \mathfrak{h}.$$

Hence ad x acts nilpotently on $\mathfrak{h}$ for all $x \in \mathfrak{h}$.

Hence by Engel's theorem $\mathfrak{h}$ is nilpotent and hence is a CSA.

Suppose $\mathfrak{h}$ is a CSA. Since $\mathfrak{h}$ is nilpotent we have $\mathfrak{h} \subset \mathfrak{g}_0(\text{ad } x)$ $\forall\, x \in \mathfrak{h}$. Choose a minimal z.

By the Lemma 1 again we get.

$$\mathfrak{g}_0(\text{ad } z) \subset \mathfrak{g}_0(\text{ad } x) \quad \forall\, x \in \mathfrak{h}.$$

Thus $\mathfrak{h}$ acts nilpotently on $\mathfrak{g}_0(\text{ad } z)/\mathfrak{h}$. If this space is nonzero we can find a non-zero common eigen vector with eigen value 0. by Engel's theorem

This means that there is $y \notin \mathfrak{h}$ with $[y, \mathfrak{h}] \subset \mathfrak{h}$. $\Rightarrow \Leftarrow$.

Lemma : 1) If $\phi : \mathfrak{g} \twoheadrightarrow \mathfrak{g}'$ is a surjective homomorphism and $\mathfrak{h}$ is a CSA, then $\phi(\mathfrak{h})$ is a CSA.

2) If $\phi : \mathfrak{g} \twoheadrightarrow \mathfrak{g}'$ is surjective $\mathfrak{h}'$ a CSA of $\mathfrak{g}'$ then any CSA $\mathfrak{h}$ of $\mathfrak{m} := \phi^{-1}(\mathfrak{h}')$ is a CSA of $\mathfrak{g}$.

Now conjugacy of BSA / CSA.

Case I Solvable Case Nothing to show for Borel.

We must prove conjugacy for Cartan subalg. In case $\mathfrak{g}$ is nilpotent.

we know any CSA is all of $\mathfrak{g}$ since $\mathfrak{g} = \mathfrak{g}_0(\mathrm{ad}\, z)$ for any $z \in \mathfrak{g}$.

So we may induct on $\dim \mathfrak{g}$. and $\mathfrak{g}$ is solvable.

Choose a abelian ideal of smallest positive dimension $\mathfrak{g}' = \mathfrak{g}/\mathfrak{a}$.

But Lemma $\mathfrak{h}_1' = [\mathfrak{h}_1]$ $\mathfrak{h}_2' = [\mathfrak{h}_2]$ are CSA.

Hence of $\mathfrak{g}'$. and hence there is a $\sigma' \in \mathcal{E}(\mathfrak{g}')$ such that $\sigma'(\mathfrak{h}_1') = \mathfrak{h}_2'$.

Lemma : $\phi : \mathfrak{g} \to \mathfrak{g}'$ If $\sigma' \in \mathcal{E}(\mathfrak{g}')$ then $\exists\, \sigma \in \mathcal{E}(\mathfrak{g})$ such that

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\phi} & \mathfrak{g}' \\ \sigma \downarrow & & \downarrow \sigma' \\ \mathfrak{g} & \xrightarrow[\phi]{} & \mathfrak{g}' \end{array}$$