Decomposition of Conformal Blocks

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Abstract

In the Masters Project we give a complete proof of the *Factorization theorem* in Conformal Field Theory and Local freeness of the sheaf of conformal blocks. We apply these results to give a proof of the *Verlinde formula*. Except for the part on Verlinde formula (Chapter 8), we mostly follow the description given in [TUY]. For the Verlinde formula, we follow the article [Beau]. We do not claim any originality of the work. We have attempted to elaborate the existing proofs mainly due to Tsuchiya, Ueno and Yamada.

Contents

Introduction

This is my master's project report presented to the University of North Carolina at Chapel Hill. We have tried to give a self contained proof of the *Factorization theorem* of conformal blocks and local freeness of the sheaf of conformal blocks. We mostly follow the description given in [TUY]. Our goal is to elaborate on their proofs and improve the clarity of the above mentioned papers. We later apply the above mentioned theorem to derive the Verlinde formula. We now discuss and give a broad overview of the main results we considered.

We start with a compact Riemann surface C of genus g and distinct marked points $Q_1, Q_2, \cdots, Q_N$ on C. The number of points is chosen such that the automorphism group of the curve is finite. To the N-pointed curve C we associate the data $\mathfrak{X} = (C; Q_1, Q_2, \cdots, Q_N; \eta_1, \eta_2 \cdots, \eta_N)$, where η_j are formal coordinates at the point Q_j. For details we refer to section 1.1.2.

Now let $\mathfrak{g}$ be a complex simple Lie algebra and $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_N) \in P_\ell^N$ be the representations attached to the marked points on C. We associate certain finite dimensional vector space (*space of covacua or conformal blocks*) $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ to $\mathfrak{X}$ and $\vec{\lambda}$. Details of this construction is given in section 3.1. These conformal blocks are basic objects of study in *Rational Conformal Field Theory.* The conformal blocks also have a nice interpretation in the settings of algebraic geometry as the spaces of regular sections ("generalized theta functions") of a line bundle on the moduli space $\mathfrak{M}$ of semistable principal G-bundles on C; where G is a connected, simply connected, affine, simple algebraic group over $\mathbb{C}$. This interpretation has been worked out in [Kum-3], also in [Fal] and for the case $G = SL_N$, we refer to [BL]. We restrict our discussion to the space of conformal blocks.

We add a point P (distinct from each Q_i) to the curve C and take the trivial representation corresponding to P. The data associated to the $(N+1)$-pointed curve C is denoted by $\widetilde{\mathfrak{X}} = (C; Q_1, Q_2, \cdots, Q_N, P; \eta_1, \eta_2 \cdots, \eta_N, \eta_{N+1})$, where η_{N+1} is a formal coordinate at P. Then there is a natural isomorphism $\mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$. This is known as *Propagation of Vacua.* For details we refer to section 3.2.

The moduli space $M_{g,N}$ parameterizes projective nonsingular curves of genus g together with N-distinct marked points. $M_{g,N}$ has a compactification $\overline{M}_{g,N}$ whose points correspond to N-pointed curves C with at most nodal singularities. The N marked points on C should be non-singular and C satisfies some stability condition equivalent to the finiteness of the automorphism group of C. The space of conformal blocks $\bigsqcup_{\mathfrak{X} \in M_{g,N}} \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ form a vector bundle $\mathcal{V}^\dagger_{\vec{\lambda}}$ on the moduli space $M_{g,N}$. It is important to study whether the vector bundle extends to a vector bundle over $\overline{M}_{g,N}$. Or in other words it is interesting to see how conformal blocks behave when the Riemann surface C degenerates to a curve with nodes. So we need to define the data $\mathfrak{X}$ and conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$ for nodal curves. For details we refer to sections 1.2 and 3.1. It turns out that the vector bundle $\mathcal{V}^{\dagger}_{\vec{\lambda}}$ extends to a vector bundle on $\overline{M}_{g,N}$. We denote it also by $\mathcal{V}^{\dagger}_{\vec{\lambda}}$. For a proof we refer to chapter 7.

Let C be a curve of genus g with a single node P. Associated to C is its normalization $\widetilde{C}$ and we have a map $\nu : \widetilde{C} \longrightarrow C$ such that $\nu^{-1}(P) = \{P', P''\}$. We have described normalization and its important properties in section 1.1.1. Corresponding to the curve $\widetilde{C}$ and formal coordinates η' and η'' at the points P' and P'' respectively, we have the data $\widetilde{\mathfrak{X}} = (\widetilde{C}; P', P'', Q_1, Q_2, \cdots, Q_N; \eta', \eta'', \eta_1, \eta_2 \cdots, \eta_N)$. The conformal blocks associated to C is intimately connected with the conformal blocks associated to $\widetilde{C}$. There is a canonical isomorphism $\bigoplus_{\mu \in P_\ell} \mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{X}}) \simeq \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$. This is known as *Factorization theorem.* It gives a decomposition of a conformal block on a genus g curve into conformal blocks on curves of lower genus. We give a complete proof in chapter 5.

Why is the above decomposition of conformal block important? Suppose we are interested in finding the dimension of conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$ associated to a curve C of genus $g > 0$. Since $\mathcal{V}^{\dagger}_{\vec{\lambda}}$ extends to a vector bundle on $\overline{M}_{g,N}$, the rank is constant. If C is smooth, we degenerate it to a nodal curve C' of genus g. Since $\mathcal{V}^{\dagger}_{\vec{\lambda}}$ is a vector bundle on $\overline{M}_{g,N}$, the dimension of the conformal block on C is the same as the dimension of the conformal block on C'. Now we can apply the factorization theorem and reduce our calculation over the normalization of C' which is a curve of lower genus. We can repeat this process and ultimately everything boils down to a computation on the projective line $\mathbb{P}^1$.

The *Verlinde formula* is an explicit formula for computing the dimension of the conformal blocks. The calculations rely on the factorization theorem and the fact that $\mathcal{V}^{\dagger}_{\vec{\lambda}}$ is a vector bundle on $\overline{M}_{g,N}$. The next question is, how do we determine the dimension of the conformal blocks on the projective line $\mathbb{P}^1$? Even for a projective line with three marked points, the dimension of the conformal block is not easy to calculate. It turns out that an explicit knowledge of the fusion ring $\mathcal{R}_\ell(\mathfrak{g})$ is required to complete the proof of Verlinde formula. G. Faltings gave a complete proof of the Verlinde formula for the classical Lie algebras and G_2. The other cases were proved by C. Teleman. For details, refer to section 8.3.

The organization of the project report is as follows:

In chapter 1, we introduce some notation and talk about the data associated with a N-pointed curve. We also describe the normalization of a nodal curve and derive the genus formula for a connected nodal curve C. The second section in chapter 1 introduces the sheaf of Kahler differentials and the dualizing sheaf of a N-pointed semistable curve. We prove a version of *Serre duality* which is relevant to our use.

Next in chapter 2, we talk about the representation theory of an affine Lie algebra $\widehat{\mathfrak{g}}$. We give a complete proof of the classification theorem of irreducible highest weight integrable $\widehat{\mathfrak{g}}$-modules. Towards the end we introduce the *energy momentum tensor* and introduce the *Virasoro operator* L_n and give a proof that L_n's form a Virasoro algebra. We close this chapter by giving a filtration on $\mathcal{H}_\lambda$ using L_0.

We then move on to define the space of vacua in chapter 3, and give a proof of the *Propagation of Vacua.* We include a much simpler proof (theorem 7) of the finite dimensionality of conformal blocks. lemma 7 clarifies the original proof of *Propagation of Vacua* in [TUY]. We have elaborated the proof of propagation of vacua, especially the "inductive steps using gauge symmetry".

In chapter 4, we introduce the correlation function of currents and prove some interesting results about a meromorphic form associated to the correlation function of currents. The correlation function of current is very important to us and is one of the important tools to analyze the decomposition of conformal blocks apart from *gauge symmetry.* We include some computations with the correlation function of currents for N-pointed projective line $\mathbb{P}^1$ and 1-pointed elliptic curve. We clarify some of the computations in [U1] and [U2]. Chapter 4 also sets up the necessary notations for the proof of the factorization theorem which we give in chapter 5. We have attempted to supply the missing details and of the proof of the factorization theorem in [TUY], [U1] and [U2].

Chapter 6 discusses the Kodaira-Spencer mapping for smooth and versal families of N-pointed stable curves of genus g. We state without proofs some important theorems in algebraic geometry and complex deformation theory. We use them in chapter 7 to give a proof of the local freeness of the sheaf of conformal blocks. In chapter 7 we also introduce the sewing construction of Ueno. We also clarify some of the lemmas in [TUY], [U1] and [U2] which leads us to theorem 23. We do not proof a convergence result of the formal construction of sections by sewing.

Finally in chapter 8, we develop the necessary background for analyzing the fusion ring $\mathcal{R}_\ell(\mathfrak{g})$. We state some results from [Fal], [Beau] and [Tel] to give a complete description of the characters of $\mathcal{R}_\ell(\mathfrak{g})$. We end by giving a proof of the Verlinde formula.

Chapter 1

Stable Pointed Curves

1.1 Stable N-pointed curves

In this section we define the stable N-pointed curves and give some examples.

1.1.1 Nodal curves and normalization

By a *nodal curve* C, we mean an algebraic curve whose singular points are nodes. Analytically around the singular points there is a neighborhood U, where the curve is given by $U_\epsilon = \{(z, w)|zw = 0\}$.

If P be a node of a nodal curve C, then we can desingularize the node by separating $U_+ = U_\epsilon \cap \{w = 0\}$ and $U_- = U_\epsilon \cap \{z = 0\}$. Thus P breaks up into two points P_+ and P_-. The new curve $\widetilde{C}$ obtained by desingularizing the nodes is called the normalization at the point P. Topologically algebraic curves can be thought of as a ramified cover π of $\mathbb{P}^1$. By normalization, we are completing the cover by removing the ramification point and adding more points. We start with a neighborhood of $\pi(P)$ which is evenly covered outside the point $\pi(P)$. If we remove the singular point P of the curve C and also the point $\pi(P)$ on the base, then the evenly covered neighborhood of P looks like $D^1 - \{O\}$. We take all the punctured disc in the inverse image of the $\pi(P)$ and add a point to each of them to get an unramified cover at that point. We can put a structure of an algebraic variety to the new curve $\widetilde{C}$ obtained after normalization. There is a natural holomorphic mapping $\nu : \widetilde{C} \to C$, such that it is identity on $\widetilde{C} - \{P_+, P_-\}$ to $C - P$ and $\nu^{-1}(P) = \{P_+, P_-\}$.

We can reverse our construction and degenerate a smooth curve to get a singular curve. We start with a compact Riemann surface R and take two distinct point P_+ and P_- on it. We choose coordinates z and w of P_+ and P_- respectively with center P_+ and P_-. Let us identify the point P_+ and P_- and obtain a new curve C. In a neighborhood of P, C is given by $zw = 0$. We can identify $C - P$ with $R - \{P_+, P_-\}$. The new curve C is a nodal curve. Thus we obtain a nodal curve from a compact Riemann surface R.

1.1.2 Stable N-pointed curves

We define stable N-pointed curves as follows:

Definition: The data $\mathfrak{X} = (C; Q_1, Q_2, \cdots, Q_N)$ consisting of a curve C and points $Q_1, Q_2, \cdots, Q_N$ on C is called a *stable N pointed curve*, if the following conditions are satisfied:

(1) The curve C is a compact Riemann surface or a nodal curve.

(2) $Q_1, Q_2, \cdots, Q_N$ are nonsingular points of the curve C.

(3) If an irreducible component of the curve C_i is a projective line i.e. $\mathbb{P}^1$, then the sum of the number of intersection points of C_i with other components and the number of Q_j's is at least three. If C_i is a rational curve with one double point or an elliptic curve, then the sum of the number of intersection points of C_i with the other components and the number of Q_j should be at least one.

(4) $\dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) = g$.

The condition (3) is equivalent to saying that $\mathrm{Aut}(\mathfrak{X})$ is a finite group, so that $\mathfrak{X}$ does not have infinitesimal automorphisms. We often add one more condition.

(5) Each component C_i must have at least one Q_j.

We give some examples of stable N-pointed curves.

Examples:

- For $g = 0$, we have $N = 3$. Any 3-pointed stable curve looks like Fig-1 in Figure 1.1.
- Any 4-pointed stable curve of genus 0 is isomorphic to one of the following in Fig-2 and Fig-3 in Figure 1.1.
- Any one pointed curve of genus 1 is isomorphic to one of the following in Figure 1.2.

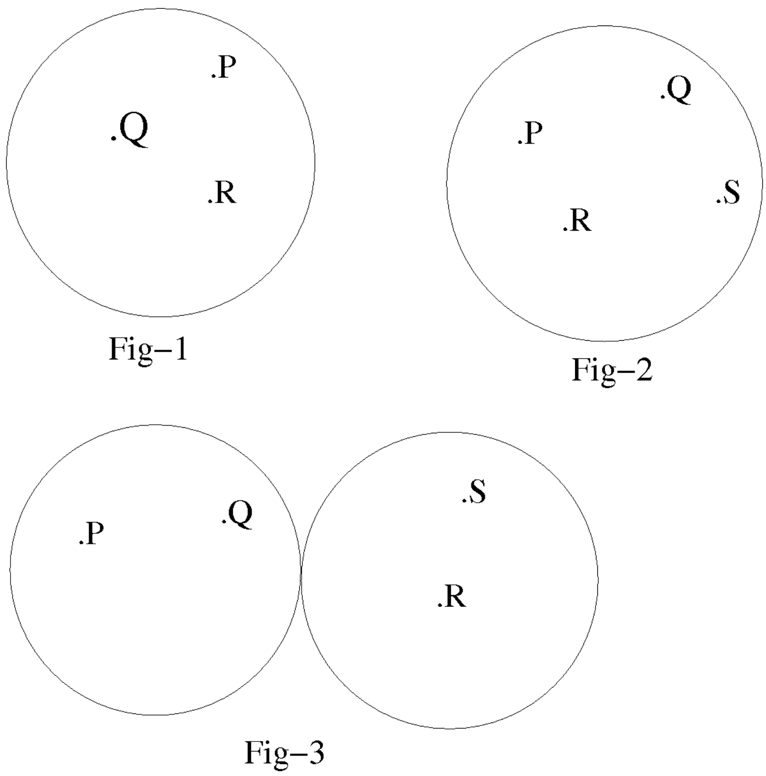

Figure 1.1: Stable 3 and 4 pointed curves of genus 0.

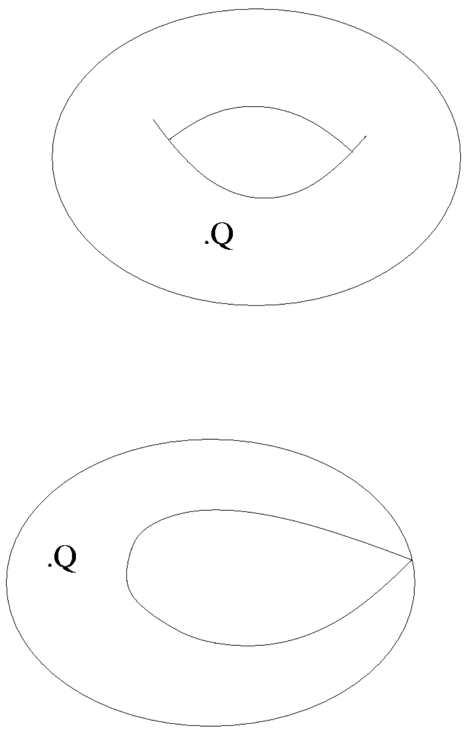

Figure 1.2: Stable 1-pointed curves of genus 1.

We give the genus formula for nodal curves.

Proposition 1. *Let C be a connected nodal curve. $P_1, P_2, \cdots, P_q$ be the nodes of the curve C. $C_1, C_2, \cdots, C_m$ be the irreducible components of geometric genus $g_1, g_2, \cdots, g_m$. Then the arithmetic genus g of C defined by $\dim_{\mathbb{C}} H^1(C, \mathcal{O}_C)$ is given as,*

$$g = \sum_{i=1}^{m}(g_i - 1) + q + 1 = \sum_{i=1}^{m} g_i + q - m + 1. \tag{1.1}$$

Proof. Let $\widetilde{C}_i$ be the normalization of C_i. By abuse of notation we write,

$$\mathcal{O}_{\widetilde{C}} = \sum_{i=1}^{m} \mathcal{O}_{\widetilde{C}_i}.$$

Then we have a exact sequence of sheaves given by,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow^{i} \mathcal{O}_{\widetilde{C}} \longrightarrow^{r} \sum_{j=1}^{q} \mathbb{C}_{P_i} \longrightarrow 0,$$

where the map r is given by $f(P_{i,+}) - f(P_{i,-})$. We take the corresponding long exact sequence in cohomology.

$$\begin{aligned} 0 \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow^{i} H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) &\longrightarrow^{r} \sum_{j=1}^{q} \mathbb{C}_{P_i} \\ &\longrightarrow^{\delta} H^1(C, \mathcal{O}_C) \longrightarrow H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) \longrightarrow 0. \end{aligned}$$

We get,

$$\begin{aligned} \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) &= \dim_{\mathbb{C}} H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) + q - \dim_{\mathbb{C}} H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) + \dim_{\mathbb{C}} H^0(C, \mathcal{O}_C) \\ &= \sum_{i=1}^{m} g_i + q - m + 1. \end{aligned}$$

□

Next we define *n-th infinitesimal neighborhood* $s^{(n)}$ of C at a point Q.

Definition Let C be a curve and Q be a nonsingular point on C. We define the *n-th infinitesimal neighborhood* $s^{(n)}$ of C at the point Q to be the $\mathbb{C}$-algebra isomorphism

$$s^{(n)} : \mathcal{O}_{C,Q}/\mathfrak{m}_Q^{n+1} \simeq \mathbb{C}[[\xi]]/(\xi^{n+1}),$$

where $\mathfrak{m}_Q$ is the maximal ideal of $\mathcal{O}_{C,Q}$ consisting of germs of holomorphic functions vanishing at Q. So by the above isomorphism, we are considering functions which vanish at the point Q at most n times. As $n \to \infty$, we get an isomorphism

$$s^{(\infty)} : \mathcal{O}_{C,Q} \simeq \mathbb{C}[[\xi]].$$

This isomorphism is known as the formal neighborhood of C at Q. We often drop the ∞ in $s^{(\infty)}$, so a formal neighborhood is denoted by s.

Definition The data $\mathfrak{X} = (C; Q_1, Q_2, \cdots, Q_N; s_1, s_2, \cdots, s_N)$ is called a stable N-pointed curve with of genus g with formal neighborhoods if,

(1) $(C; Q_1, Q_2, \cdots, Q_N)$ is a N-pointed stable curve of genus g.

(2) s_j is a formal neighborhood of C at Q_j.

We can define a stable N-pointed curve of genus g with n-th formal neighborhood as above only replacing s_j by $s_j^{(n)}$. The data for n-th formal neighborhoods will be denoted by,

$$\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \cdots, Q_N; s_1^{(n)}, s_2^{(n)}, \cdots, s_N^{(n)}).$$

1.2 Serre Duality

In this section we talk about the dualizing sheaf of a stable N-pointed curve and state a version of *Serre Duality Theorem.* For further details we refer to [HM].

1.2.1 Residue pairing

Let C be a nodal curve and Ω^1_C be the sheaf of *Kahler differentials.* ω_C be the *dualizing sheaf* on the curve C. Near a singular point P, the curve is analytically isomorphic, or in other words locally given by the equation

$$xy = 0.$$

The sheaf of Kahler differentials Ω^1_C is expressed as,

$$\Omega^1_C = (\mathcal{O}_C dx + \mathcal{O}_C dy)/(xdx + ydy)\mathcal{O}_C.$$

The *dualizing sheaf of a nodal curve* C can be defined as a subsheaf of the pushforward of the sheaf of rational differentials on $\widetilde{C}$. It associates on an open subset U of C, the space of rational *1-forms* τ on $\nu^{-1}(U) \subset \widetilde{C}$ having at worst simple poles at the point $P_{i,+}$ and $P_{i,-}$ lying over each node P_i. Moreover for each such pair of point we have,

$$\operatorname*{Res}_{P_{i,+}} \tau + \operatorname*{Res}_{P_{i,-}} \tau = 0.$$

Let $\nu : \widetilde{C} \to C$ be the normalization of$C$. $P_1, P_2, \cdots, P_q$ be the set of double points of the curve C. For each of these double points, put $\nu^{-1}(P_i) = \{P_{i,+}, P_{i,-}\}$. Then,

$$0 \to \omega_C \to \omega_{\widetilde{C}}(\sum_i^q (P_{i,+} + P_{i,-})) \to^r \bigoplus_{i=1}^q \mathbb{C} \to 0,$$

where at each double point the mapping r is given by the sum of the residues at $P_{i,+}$ and $P_{i,-}$ i.e.

$$\operatorname*{Res}_{P_{i,+}}(\tau) + \operatorname*{Res}_{P_{i,-}}(\tau).$$

Our discussion can be summarized in the form of the following Lemma:

Lemma 1. *Let τ be a holomorphic section of the dualizing sheaf ω_C. We consider τ as a rational section of meromorphic 1-forms on the normalization $\widetilde{C}$ with at worst simple poles at the points $P_{i,+}$ and $P_{i,-}$. Then the sum of the residues of τ is zero.*

1.2.2 Serre duality

Our next lemma is about the Laurent expansion of global meromorphic functions on the stable N-pointed curves.

Lemma 2. *Let $\mathfrak{X} = (C; Q_1, Q_2, \cdots, Q_N; s_1, s_2, \cdots, s_N)$ be a stable N-pointed curve of genus g with formal neighborhoods satisfying the condition* (5) *of the definition in section 1.1.2. By t_j, we denote the Laurent expansions at Q_j with respect to a formal parameter $\xi_j = s_j^{-1}(\xi)$. Then the following homomorphisms are injective:*

$$t = \oplus t_j : H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j)) \longrightarrow \bigoplus_{j=1}^{N} \mathbb{C}((\xi_j)),$$
$$t = \oplus t_j : H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j)) \longrightarrow \bigoplus_{j=1}^{N} \mathbb{C}((\xi_j))d\xi_j;$$

where ω_C is the dualizing sheaf of the curve C and

$$H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j)) = \lim_{n\to\infty} H^0(C, \mathcal{O}_C(n\sum_{j=1}^{N} Q_j)),$$

$$H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j)) = \lim_{n\to\infty} H^0(C, \omega_C(n\sum_{j=1}^{N} Q_j)).$$

Proof. If $t_j(f) = 0$, for an element $f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$, then it vanishes on the whole component of the curve. By condition (5) of the definition in section 1.1.2, the existence of Q_j on each components forces the function to vanish on every component. Thus $f = 0$. So the mapping t_j is injective. A similar proof follows for $\omega \in H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j))$. □

By the above lemma we can consider $H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$ and $H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j))$ as a subspace of $\bigoplus_{j=1}^{N} \mathbb{C}((\xi_j))$ and $\bigoplus_{j=1}^{N} \mathbb{C}((\xi_j))d\xi_j$ respectively. There is a natural residue pairing between $\bigoplus_{j=1}^N \mathbb{C}((\xi_j))$ and $\bigoplus_{j=1}^N \mathbb{C}((\xi_j))d\xi_j$ given by,

$$\langle\ ,\ \rangle : \bigoplus_{j=1}^N \mathbb{C}((\xi_j)) \times \bigoplus_{j=1}^N \mathbb{C}((\xi_j))d\xi_j \longrightarrow \mathbb{C}.$$

$$\langle f_1(\xi_1), \cdots, f_N(\xi_N); g_1(\xi_1)d\xi_1, \cdots, g_N(\xi_N)d\xi_N\rangle \longrightarrow \sum_{j=1}^N \operatorname*{Res}_{\xi_j=0}(f_j(\xi_j)g_j(\xi_j)d\xi_j).$$

We can introduce the same pairing between

$$H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j)) \times H^0(C, \omega_C(*\sum_{j=1}^N Q_j)) \longrightarrow \mathbb{C}.$$

Theorem 1. *Under the above pairing, the vector space $H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j))$ and the vector space $H^0(C, \omega_C(*\sum_{j=1}^N Q_j))$ are annihilators of each other.*

Proof. We know that the residue is independent of the choice of local coordinates. So we can do the computation using the formal coordinates at the point Q_j. For any positive integers m and n, we have the following exact sequence of sheaves:

$$0 \to \mathcal{O}_C(-m\sum_{j=1}^N Q_j) \to \mathcal{O}_C(n\sum_{j=1}^N Q_j) \to \bigoplus_{j=1}^N \bigoplus_{k=-n}^{m-1} \mathbb{C}{\xi_j}^k \to 0.$$

From the long exact sequence in cohomology,

$$0 \to H^0(C, \mathcal{O}_C(-m\sum_{j=1}^N Q_j)) \to H^0(C, \mathcal{O}_C(n\sum_{j=1}^N Q_j)) \to^p \bigoplus_{j=1}^N \bigoplus_{k=-n}^{m-1} \mathbb{C}{\xi_j}^k$$

$$\to^c H^1(C, \mathcal{O}_C(-m\sum_{j=1}^N Q_j) \to H^1(C, \mathcal{O}_C(n\sum_{j=1}^N Q_j)) \to 0.$$

Now by the usual Serre duality theorem (see [HM] for reference) for stable N-pointed curves, we get $H^0(C, \omega_C(m\sum_{j=1}^N Q_j))$ and $H^1(C, \mathcal{O}_C(-m\sum_{j=1}^N Q_j))$ are dual to each other under the residue pairing. We will prove that on the image of c the pairing is given by,

$$\langle\ ,\ \rangle : H^1(C, \mathcal{O}_C(-m\sum_{j=1}^N Q_j)) \times H^0(C, \omega_C(m(\sum_{j=1}^N Q_j)) \longrightarrow \mathbb{C}.$$

$$\langle c(\oplus_{j=1}^N g_j(\xi_j)), \tau))\rangle = \sum_{j=1}^N \operatorname*{Res}_{\xi_j=0} g_j(\xi_j)\tau_j, \tag{1.2}$$

where $g_j(\xi_j)$ is an element of $\oplus_{k=-n}^{m-1}\mathbb{C}{\xi_j}^k$ and τ is an element of $H^0(C, \omega_C(m\sum_{j=1}^N Q_j))$. τ_j is the Laurent expansion τ about the point Q_j in formal coordinates ξ_j. It is known for compact Riemann surfaces but we will extend it to nodal curves. For the moment we assume (1.2) and complete the proof of the theorem.

Now $\oplus_{j=1}^{N} g_j(\xi_j)$ is in the image of $H^0(C, \mathcal{O}_C(n\sum_{j=1}^{N} Q_j))$ if and only if

$$\sum_{j=1}^{N} \operatorname*{Res}_{\xi_j=0}(g_j(\xi_j)\tau_j) = 0,$$

for any $\tau \in H^0(C, \omega_C(m\sum_{j=1}^{N} Q_j))$. This follows from the above long exact sequence in cohomology, $\mathrm{Ker}(c) = \mathrm{Im}(p)$ and Serre Duality. We will use this in the proof.

Assume that under the residue pairing $\oplus_{j=1}^{N} f_j(\xi_j) \in \oplus_{j=1}^{N} \mathbb{C}((\xi_j))$ is annihilated by $H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j))$. We want to show that $\oplus_{j=1}^{N} f_j(\xi_j) \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$.

We fix a positive integer m and consider $\oplus_{j=1}^{N}(f_j(\xi_j)) \mod \xi_j^m$. That is if

$$f_j(\xi_j) = \sum_{k=-n_j}^{\infty} a_{j,k}{\xi_j}^k,$$

then we define

$$f_{j,m}(\xi_j) = \sum_{k=-n_j}^{m-1} a_{j,k}{\xi_j}^k.$$

Now since $\oplus_{j=1}^{N} f_j(\xi_j)$ is annihilated by $H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j))$, so $\oplus_{j=1}^{N} f_{j,m}(\xi_j)$ is annihilated by $H^0(C, \omega_C(m\sum_{j=1}^{N} Q_j))$. We choose $n = max(n_j)$ and by our previous discussion, we get that $f_{j,m}$ comes form an element $f^{(m)}$ of $H^0(C, \mathcal{O}_C(n\sum_{j=1}^{N} Q_j))$. We claim that if we choose our m big enough, the element $f^{(m)}$ is same as $\oplus_{j=1}^{N} f_j(\xi_j)$.

Let $f^{(m)}(\xi_j)$ be the Laurent series expansion of $f^{(m)}$ around Q_j with formal parameter ξ_j. Suppose $f^{(m)}(\xi_j)$ and $f_j(\xi_j)$ are not the same, we consider the difference $g_j(\xi_j)$. We have

$$g_j(\xi_j) = f_j(\xi_j) - f^m(\xi_j) \equiv 0 \mod ({\xi_j}^m).$$

If all the g_j's are zero, then $f^{(m)}$ and $\oplus_{j=1}^{N} f_j(\xi_j)$ are same. So without loss of generality, we can assume that $g_k(\xi_k)$ is not zero. Let $s \geq m$ and

$$g_k(\xi_k) = \sum_{i=0}^{\infty} b_{s+i}{\xi_k}^{s+i}.$$

Also assume that $b_s \neq 0$. Now we know that $\oplus_{j=1}^{N} f_j(\xi_j)$ and $f^{(m)}$ are both annihilated by $H^0(C, \omega_C(m\sum_{j=1}^{N} Q_j))$. So $\oplus_{j=1}^{N} g_j(\xi_j)$ is also annihilated by $H^0(C, \omega_C(m\sum_{j=1}^{N} Q_j))$.

Now we choose m large enough such that $H^0(C, \omega_C((s+1)Q_k)))$ has a nontrivial element. Take $\omega \in H^0(C, \omega_C((s+1)Q_k)))$. We now compute the residue pairing of ω and $\oplus_{j=1}^{N} g_j(\xi_j)$. We get,

$$\sum_{j=1}^{N} \operatorname*{Res}_{\xi_j=0} g_j(\xi_j)\omega_j(\xi_j) = \operatorname*{Res}_{\xi_k=0} g_k\omega_k(\xi_k) = b_s \neq 0,$$

where $\omega_j(\xi_j)$ is the Laurent expansion of ω around Q_j. This is a contradiction to the fact that $\oplus_{i=1}^{N} g_j(\xi_j)$ is annihilated by $H^0(C, \omega_C(m\sum_{j=1}^{N} Q_j))$. This completes the proof for the functions. We now prove (1.2).

Given a nodal curve C, we have a natural smooth curve associated to it namely its normalization $\widetilde{C}$. We start with the usual residue on $\widetilde{C}$ and extend it to nodal curves. For simplicity assume C is a nodal curve with only one node at P. Let $\nu : \widetilde{C} \to C$ be the normalization map. We get $\nu^*\omega_C = \omega_{\widetilde{C}}(P_+ + P_-)$. Now $\nu^*\omega_C = \omega_{\widetilde{C}}(P_+ + P_-)$. This implies that for m large enough,

$$H^0(\widetilde{C}, \omega_{\widetilde{C}}(m\sum_{j=1}^{N} Q_j)) \subset \nu^* H^0(C, \omega_C(m\sum_{j=1}^{N} Q_j)).$$

Now if $\oplus_{j=}^{N} g_j(\xi_j)$ is annihilated by $H^0(C, \omega_C(m\sum_{j=1}^{N} Q_j))$, this implies that $\oplus_{j=1}^{N} g_j(\xi_j)$ is annihilated by $H^0(\widetilde{C}, \omega_{\widetilde{C}}(m\sum_{j=1}^{N} Q_j))$. By the previous argument, we get $\oplus_{j=1}^{N} g_j(\xi_j)$ comes from an element $\widetilde{g}$ of $H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(*\sum_{j=1}^{N} Q_j))$. If we can show that $\widetilde{g}$ can be realized as an element of $H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$ we are done. So we need to show,

$$\widetilde{g}(P_+) = \widetilde{g}(P_-).$$

We choose an element τ in $\nu^* H^0(C, \omega_C(m\sum_{j=1}^{N} Q_j)) \setminus H^0(\widetilde{C}, \omega_{\widetilde{C}}(m\sum_{j=1}^{N} Q_j))$. A large enough m will guarantee the existence of such an element τ. Since the sum of the residues of a meromorphic 1-form is zero,

$$\sum_{j=1}^{N} \operatorname*{Res}_{Q_j}(\widetilde{g}\tau) = 0.$$

Now τ has a pole of order 1 at the point P_+ and at P_-. Now $\widetilde{g}$ is holomorphic at P_+ and also at P_-. So

$$\begin{aligned} \operatorname*{Res}_{P_+} \widetilde{g}\tau + \operatorname*{Res}_{P_-} \widetilde{g}\tau &= 0, \\ \widetilde{g}(P_+) \operatorname*{Res}_{P_+} \tau + \widetilde{g}(P_-) \operatorname*{Res}_{P_-} \tau &= 0. \end{aligned}$$

Now since τ comes from a section of ω_C, we know that

$$\operatorname*{Res}_{P_+} \tau + \operatorname*{Res}_{P_-} \tau = 0.$$

This implies that $\widetilde{g}(P_+) = \widetilde{g}(P_-)$. Thus $\widetilde{g}$ can be realized in $H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$.

The proof for $H^0(C, \omega_C(*\sum_{j=1}^{N} Q_j))$ relies on the same argument. We describe it below. For every m and n, we have an exact sequence of sheaves given by,

$$0 \to \omega_C(-m\sum_{j=1}^{N} Q_j) \to \omega_C(n\sum_{j=1}^{N} Q_j) \to \bigoplus_{j=1}^{N} \bigoplus_{k=-n}^{m-1} \mathbb{C}{\xi_j}^k d\xi_j \to 0.$$

This gives rise to a long exact sequence is cohomology.

$$\begin{aligned} 0 \to H^0(C, \omega_C(-m\sum_{j=1}^{N} Q_j)) \to H^0(C, \omega_C(n\sum_{j=1}^{N} Q_j)) \to^p \bigoplus_{j=1}^{N} \bigoplus_{k=-n}^{m-1} \mathbb{C}{\xi_j}^k d\xi_j \\ \to^c H^1(C, \omega_C(-m\sum_{j=1}^{N} Q_j)) \to H^1(C, \omega_C(n\sum_{j=1}^{N} Q_j)) \to 0. \end{aligned}$$

We know by Serre duality, that the dual of $H^1(C, \omega_C(-m\sum_{j=1}^N Q_j))$ is isomorphic with $H^0(C, \mathcal{O}_C(m\sum_{j=1}^N Q_j))$. Thus $\oplus_{j=1}^N h_j(\xi_j)d\xi_j \in \bigoplus_{j=1}^N \bigoplus_{k=-n}^{m-1} \mathbb{C}{\xi_j}^k d\xi_j$ is annihilated by $H^0(C, \mathcal{O}_C(m\sum_{j=1}^N Q_j))$ if and only if it is the image of an element of $H^0(C, \omega_C(n\sum_{j=1}^N Q_j))$ under the map p. Now we assume that $\oplus_{j=1}^N h_j(\xi_j)d\xi_j$ is annihilated by $H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j))$. As before we fix a positive integer m and n consider $\oplus_{j=1}^N (h_j(\xi_j d\xi_j)) \mod \xi_j^m$. That is if

$$h_j(\xi_j)d\xi_j = \sum_{k=-n_j}^{\infty} a_{j,k}{\xi_j}^k d\xi_j,$$

we define $h_{j,m}(\xi_j)d\xi_j = \sum_{k=-n_j}^{m-1} a_{j,k}{\xi_j}^k d\xi_j$.

Since $\oplus_{j=1}^N h_j(\xi_j)d\xi_j$ is annihilated by $H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j))$, we get $\oplus_{j=1}^N h_{j,m}(\xi_j)d\xi_j$ is annihilated by $H^0(C, \mathcal{O}_C(m\sum_{j=1}^N Q_j))$. Choose $n = max(n_j)$. Thus by our previous discussion, we get that $\oplus_{j=1}^N h_{j,m}d\xi_j$ comes from an element $h^{(m)}$ of $H^0(C, \omega_C(n\sum_{j=1}^N Q_j))$. Now if we choose our m big enough, the element $h^{(m)}$ is same as $\oplus_{j=1}^N h_j(\xi_j)d\xi_j$.

Let $h^{(m)}(\xi_j)d\xi_j$ be the Laurent expansion of $h^{(m)}$ around Q_j . Suppose $h^{(m)}(\xi_j)d\xi_j$ and $h_j(\xi_j)d\xi_j$ are not equal. We consider their difference $g_j(\xi_j)$. Now

$$g_j(\xi_j)d\xi_j = h_j(\xi_j)d\xi_j - h^{(m)}(\xi_j)d\xi_j \equiv 0 \mod ({\xi_j}^m).$$

If all the g_j's are zero, then $h^{(m)}$ and $\oplus_{j=1}^N h_j(\xi_j)$ are same. So without loss of generality we can assume that $g_k(\xi_k)$ is not zero. Let $s \geq m$.

$$g_k(\xi_k)d\xi_j = \sum_{i=0}^{\infty} b_{s+i}{\xi_k}^{s+i} d\xi_j.$$

Assume that $b_s \neq 0$. Now we know that $\oplus_{j=1}^N h_j(\xi_j)d\xi_j$ and $h^{(m)}$ are both annihilated by $H^0(C, \mathcal{O}_C(m\sum_{j=1}^N Q_j))$. Thus $\oplus_{j=1}^N g_j(\xi_j)d\xi_j$ is also annihilated by $H^0(C, \mathcal{O}_C(m\sum_{j=1}^N Q_j))$. Choose m large enough such that $H^0(C, \mathcal{O}_C((s+1)Q_k)))$ has a nontrivial element f. We now compute the residue pairing of f and $\oplus_{j=1}^N g_j\xi_j d\xi_j$. We get,

$$\sum_{j=1}^N \operatorname*{Res}_{\xi_j=0} f_j(\xi)g_j(\xi_j)d\xi_j = \mathrm{Res}_{\xi_k=0} f_k(\xi_k)g_k(\xi_k) = b_s \neq 0;$$

where $f_j(\xi_j)$ is the Laurent expansion of f around Q_j. This is a contradiction to the fact that $\oplus_{i=1}^N g_j(\xi_j)d\xi_j$ is annihilated by $H^0(C, \omega_C(m\sum_{j=1}^N Q_j))$. This completes the entire proof. □

Chapter 2

Affine Lie algebras and Highest Weight Integrable Modules

2.1 Representations of complex semi-simple Lie algebras

In this section we recollect some theorems about representations of complex semi-simple Lie algebras.

2.1.1 Complex simple Lie algebras

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. We know that all linear representations (finite dimensional) of $\mathfrak{g}$ are completely reducible. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{b} \supset \mathfrak{h}$ be a Borel subalgebra. Let Δ be a root system of $(\mathfrak{g}, \mathfrak{h})$. $\Delta = \Delta^+ \sqcup \Delta^-$ be the decomposition of the root system into positive and negative roots. We know,

$$\Delta^- = -\Delta^+.$$

$\mathfrak{h}^*_{\mathbb{R}}$ be the real span of Δ and $\mathfrak{h}^* = \mathfrak{h}^*_{\mathbb{R}} \otimes \mathbb{C}$. Let $\{\alpha_1, \cdots, \alpha_r\} \subset \Delta$ be the set of simple roots in Δ and $\{{\alpha_1}^\vee, \cdots, \alpha_r^\vee\} \subset \mathfrak{h}$ be the corresponding simple coroots, where $r = \dim_{\mathbb{C}} \mathfrak{h}$. For any $\alpha \in \Delta$, $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the root space corresponding to the root α. For each element of $t \in \mathfrak{h}$, there is an unique element $\lambda_t \in \mathfrak{h}^*$ such that:

$$\lambda_t(h) = (t, h),$$

for all $h \in \mathfrak{h}$. Using this we can introduce an inner product $(\ , \ )$ on $\mathfrak{h}^*$ given as follows:

$$(\lambda, \mu) = (t_\lambda, t_\mu),$$

where t_λ and t_μ are the unique elements of $\mathfrak{h}$ corresponding to λ and μ respectively. We define the highest root of $(\mathfrak{g}, \mathfrak{h})$ to be the root $\theta \in \Delta^+$ such that:

$$\theta + \alpha_i \notin \Delta,$$

for all simple roots α_i. For type A_r the highest root is given by,

$$\alpha_1 + \alpha_2 + \cdots + \alpha_r.$$

For type B_r it is given by,

$$\alpha_1 + 2\sum_{j=2}^{r} \alpha_j.$$

A list of the highest roots can be found in [Hum].

Let $\langle\ ,\ \rangle$ denote the normalized Cartan Killing form on $\mathfrak{g}$ such that $\langle\theta, \theta\rangle = 2$, where θ is the highest root of $(\mathfrak{g}, \mathfrak{h})$. For $\mathfrak{sl}_2(\mathbb{C})$, the normalized Cartan Killing form is given by,

$$\langle X, Y\rangle = Tr(X.Y).$$

2.1.2 Representation of semi-simple Lie algebras

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. V be a $\mathfrak{g}$-module not necessarily finite dimensional and $\omega \in \mathfrak{h}^*$ be a linear form on $\mathfrak{h}$. Let V^ω be the set of all vectors in $v \in V$ such that $H.v = \omega(H)v$, for all $H \in \mathfrak{h}$. The dimension of V^ω is defined to be the multiplicity of V^ω and ω is called a *weight* of V. A *primitive element* v of weight ω of a $\mathfrak{g}$-module V is a vector such that the line $\mathbb{C}v$ is stable under the action of the Borel subalgebra $\mathfrak{b}$. We state the following theorem from [Ser]:

Theorem 2. *Let V be an irreducible $\mathfrak{g}$-module containing a primitive element v of weight ω. Then:*

(a) v is the only primitive of V of weight ω upto scalar multiplication and ω is called the highest weight of V.

(b) The weights π of V are of the form,

$$\pi = \omega - \sum_i m_i\alpha_i$$

with $m_i \in \mathbb{N}$. They have finite multiplicity and in particular ω has multiplicity 1. One has $V = \sum V^\pi$.

(c) For two irreducible $\mathfrak{g}$-modules V_1 and V_2 with highest weights ω_1 and ω_2 to be isomorphic, it is necessary and sufficient that $\omega_1 = \omega_2$.

We state another useful theorem from [Ser].

Theorem 3. *(a) For any $\omega \in \mathfrak{h}^*$, there is an irreducible $\mathfrak{g}$-module of highest weight ω.*

(b) Let E_ω be the irreducible $\mathfrak{g}$-module with highest weight ω. For E_ω to be finite dimensional, it is necessary and sufficient that $\forall\alpha \in \Delta^+$ we have $\omega(t_\alpha) \in \mathbb{Z}^+ \cup \{0\}$, where t_α is the element corresponding to α. $\mathbb{Z}^+$ denotes the set of positive integers.

2.1.3 Casimir element of a representation

We recall the Casimir element corresponding to a representation of a semi-simple Lie algebra $\mathfrak{g}$ of dimension n. Let $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful finite dimensional representation of $\mathfrak{g}$, we define a bilinear form β on $\mathfrak{g}$ by $\beta(x, y) = Tr(\phi(x)\phi(y))$. We claim that β is nondegenerate, symmetric, $\mathfrak{g}$-invariant. Let X, Y and Z be endomorphisms of a finite dimensional vector space, then $[X, Y]Z = XYZ - YXZ$ and $X[Y, Z] = XYZ - XZY$. But $Tr(Y(XZ)) = Tr(X(ZY))$. We get,

$$Tr([X, Y]Z) = Tr(XYZ) - Tr(YXZ) = Tr(XYZ) - Tr(XZY) = Tr(X[YZ]).$$

Now to show β is $\mathfrak{g}$-invariant, we need to show $\beta([x, y], z) = \beta(x, [y, z])$. In other words $Tr([\phi(x), \phi(y)]\phi(z)) = Tr(\phi(x)[\phi(y), \phi(z)])$ which is clear from the previous identity. Thus β is $\mathfrak{g}$-invariant.

The radical of asymmetric form is defined to be $S = \{x \in \mathfrak{g} | \beta(\phi(x), \phi(y)) = 0 \, \forall y \in \mathfrak{g}\}$. Since β is $\mathfrak{g}$-invariant, the radical of β is an ideal in $\mathfrak{g}$. We can use Cartan Criterion for solvability to conclude $\phi(S)$ is solvable. But ϕ is faithful, which implies $\phi(S) \cong S$. So $S = 0$. Thus β is nondegenerate.

Let β be any nondegenerate symmetric associative bilinear form on a semisimple Lie algebra $\mathfrak{g}$ and $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be any representation. Since β is a nondegenerate bilinear form on $\mathfrak{g}$, it identifies $\mathfrak{g}$ with its dual. Let $(x_1, x_2, \cdots, x_n)$ be a basis of $\mathfrak{g}$ of dimension n. Let $(y_1, y_2, \cdots, y_n)$ be the dual basis with respect to the bilinear form β.

Suppose $[x, x_i] = \sum_{j=1}^{n} a_{ij} x_j$ and $[x, y_i] = \sum_{j=1}^{n} b_{ij} y_j$, where a_{ij} and b_{ij} are constants. Since x_i and y_i are dual to each other we can write $a_{ik} = \beta([x, x_i], y_k) = \beta(-[x_i, x], y_k) = \beta(x_i, -[x, y_k]) = -\beta(x_i, [x, y_k]) = -b_{ik}$.

Define $c_\phi(\beta) = \sum_{i=1}^{n} \phi(x_i)\phi(y_i)$. We often drop β in the notation. Now c_ϕ is an element of $\mathfrak{gl}(V)$. We compute the Lie bracket of c_ϕ with $\phi(x)$, where $x \in \mathfrak{g}$.

$$\begin{aligned}
[\phi(x), c_\phi(\beta)] &= \sum_{i=1}^{n} [\phi(x), \phi(x_i)\phi(y_i)] \\
&= \sum_{i=1}^{n} [\phi(x), \phi(x_i)]\phi(y_i) + \sum_{i=1}^{n} \phi(x_i)[\phi(x_i), \phi(y_i)] \\
&= \sum_{i=1}^{n} a_{ij}\phi(x_j)\phi(y_i) + \sum_{i=1}^{n} b_{ij}\phi(x_j)\phi(y_i) \\
&= 0.
\end{aligned}$$

Thus $c_\phi(\beta)$ is an endomorphism of the V commuting with ϕ. In particular if $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation with non degenerate trace form, then $\beta(x, y) = Tr(\phi(x)\phi(y))$. We fix a basis $(x_1, x_2, \cdots, x_n)$. We define the *Casimir element* of the representation by c_ϕ with respect to the above basis. In particular if ϕ is irreducible, then by Schur's lemma, c_ϕ is a scalar. The trace of c_ϕ is given by $Tr(c_\phi) = \sum_{i=1}^{n} Tr(\phi(x_i)\phi(y_i)) = n$. So c_ϕ is given by $\frac{Tr(c_\phi)}{dim(V)}$. Hence the Casimir element is independent of the basis.

2.2 Affine Lie algebras

$\mathbb{C}[[\xi]]$ and $\mathbb{C}((\xi))$ be the ring of formal power series and the field of formal Laurent power series in ξ respectively. Let $\mathcal{A} = \mathbb{C}[\xi, \xi^{-1}]$. We define the *affine Kac-Moody algebra* (for short the *affine Lie algebra*)

$$\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{A} \oplus \mathbb{C}c.$$

The bracket operation is given as follows:

$$[x \otimes \xi^m + \lambda.c, y \otimes \xi^n + \beta.c] = [x, y] \otimes \xi^{m+n} + c.\langle x, y\rangle \delta_{m+n,0},$$

where c is an element of center of $\widetilde{\mathfrak{g}}$; $x, y \in \mathfrak{g}$; m, $n \in \mathbb{Z}$ and $\lambda, \beta \in \mathbb{C}$. We are interested in the completion

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c.$$

The bracket operation is given as follows:

$$[x \otimes p + \lambda.c, y \otimes q + \beta.c] = [x, y] \otimes pq + c.\langle x, y\rangle \operatorname*{Res}_{\xi=0} q dp,$$

where $p, q \in \mathbb{C}((\xi))$, $x, y \in \mathfrak{g}$ and $\lambda, \beta \in \mathbb{C}$.

Clearly $\widetilde{\mathfrak{g}}$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$. The Lie algebra $\widehat{\mathfrak{g}}$ admits a derivation d given by,

$$d(c) = 0, \quad d(x \otimes p) = x \otimes \xi(\frac{dp}{d\xi}),$$

for $p \in \mathbb{C}((\xi))$ and $x \in \mathfrak{g}$. It is clear that d keep $\widetilde{\mathfrak{g}}$ stable. So we have the following semidirect products $\mathbb{C}d \ltimes \widehat{\mathfrak{g}}$ and $\mathbb{C}d \ltimes \widetilde{\mathfrak{g}}$.

Define the (formal) *Loop algebra*

$$\mathcal{L}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}((\xi)).$$

The bracket operation is defined by,

$$[x \otimes p, y \otimes q] = [x, y] \otimes pq;$$

where $x, y \in \mathfrak{g}$ and $p, q \in \mathbb{C}((\xi))$. We can view $\widehat{\mathfrak{g}}$ as a one dimensional central extension of $\mathcal{L}(\mathfrak{g})$, i.e.

$$0 \to \mathbb{C}c \to \widehat{\mathfrak{g}} \to \mathcal{L}(\mathfrak{g}) \to 0.$$

For proofs see [Kum1], [Gar].

2.2.1 Some subalgebras of $\widehat{\mathfrak{g}}$

Now $\mathfrak{g}$ is embedded in $\widehat{\mathfrak{g}}$ as a subalgebra $\mathfrak{g} \otimes \xi^0$. We define the standard *Cartan subalgebra*

$$\widehat{\mathfrak{h}} := \mathfrak{h} \otimes \xi^0 \oplus \mathbb{C}c.$$

The standard *Borel subalgebra*

$$\widehat{\mathfrak{b}} := \mathfrak{g} \otimes (\xi\mathbb{C}[[\xi]]) \oplus \mathfrak{b} \otimes \xi^0 \oplus \mathbb{C}c.$$

More generally for any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, we can define a subalgebra

$$\widehat{\mathfrak{p}} := \mathfrak{g} \otimes (\xi\mathbb{C}[[\xi]]) \oplus \mathfrak{p} \otimes \xi^0 \oplus \mathbb{C}c.$$

The subalgebras will be referred to as *standard parabolic subalgebras* of $\widehat{\mathfrak{g}}$.

For any standard parabolic $\mathfrak{p}$ of $\mathfrak{g}$, let $\mathfrak{l}$ be its unique Levi component which is stable under the adjoint action $\mathfrak{h}$ and $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{p}$. Thus we have $\mathfrak{p} = \mathfrak{u} \oplus \mathfrak{l}$. Define the following subalgebras of $\widehat{\mathfrak{p}}$:

$$\begin{aligned} \widehat{\mathfrak{l}}_{\mathfrak{p}} &:= \mathfrak{l} \otimes \xi^0 \oplus \mathbb{C}c, \\ \widehat{\mathfrak{u}}_{\mathfrak{p}} &:= \mathfrak{g} \otimes \xi\mathbb{C}[[\xi]] \oplus \mathfrak{u} \otimes \xi^0. \end{aligned}$$

In fact we have,

$$\widehat{\mathfrak{p}} := \widehat{\mathfrak{u}}_{\mathfrak{p}} \oplus \widehat{\mathfrak{l}}_{\mathfrak{p}}.$$

We also define

$$\widehat{\mathfrak{u}}_{\mathfrak{p}}^- := \xi^{-1}\mathbb{C}[\xi^{-1}] \otimes \mathfrak{g} \oplus \mathfrak{u}^{-1} \otimes \xi^0,$$

where $\mathfrak{u}^{-1}$ to be the nilradical of the opposite parabolic $\mathfrak{p}^{-1} \subset \mathfrak{g}$. We get,

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{p}} \oplus \widehat{\mathfrak{u}}_{\mathfrak{p}}^-.$$

Now at the end we define the *principal three dimensional subalgebra*

$$\mathfrak{r} := \mathfrak{g}_\theta \otimes \xi^{-1} \oplus \mathfrak{g}_{-\theta} \otimes \xi \oplus \mathbb{C}(c - \theta^\vee),$$

where $\mathfrak{g}_\theta$ is the root space corresponding to the highest root θ and $\theta^\vee \in \mathfrak{h}$ is the coroot corresponding to θ. We take $x_\theta \in \mathfrak{g}_\theta$ and $y_\theta \in \mathfrak{g}_{-\theta}$ satisfying $\langle x_\theta, y_\theta \rangle = 1$. This subalgebra $\mathfrak{r}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ under the map $X \to y_\theta \otimes \xi$, $Y \to x_\theta \otimes \xi^{-1}$ and $H \to c - \theta^\vee$.

2.2.2 Integrable, highest weight $\widehat{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}$-modules

Let $\mathfrak{s}$ be a Lie algebra and V be a $\mathfrak{s}$-module. Then V is called a *locally finite $\mathfrak{s}$ module* if, for $v \in V$ there exists a finite dimensional $\mathfrak{s}$ stable subspace V_v of V containing v.

Let $T : V \to V$ be a linear operator of vector spaces. T is said to be *locally finite* if, for every $v \in V$, there exists a T-stable subspace V_v of V containing v. T is said to be *locally nilpotent* if, for $v \in V$, there exists $n_v \in \mathbb{Z}^+$ such that $T^{n_v}(v) = 0$.

A representation $\mathcal{H}$ of $\widehat{\mathfrak{g}}$ or $\widetilde{\mathfrak{g}}$ is called *integrable*, if $\mathcal{H}$ is locally finite as a $\mathfrak{g}$-module as well as a locally finite as $\mathfrak{r}$-module. Clearly any submodule and quotient module of a integrable representation is integrable.

A representation $\mathcal{H}$ of $\widehat{\mathfrak{g}}$ is called a *highest weight integrable representation* if, $\mathcal{H}$ is a integrable $\widehat{\mathfrak{g}}$-module with highest weight vector $v_+ \in \mathcal{H}$ such that the following holds:

(a) The line $\mathbb{C}v_+$ is stable under the action of the $\widehat{\mathfrak{b}}$, and

(b) v_+ generates $\mathcal{H}$ as a $\widehat{\mathfrak{g}}$-module or in other words the only $\widehat{\mathfrak{g}}$-submodule of $\mathcal{H}$ containing v_+ is the whole $\mathcal{H}$.

Similarly it is defined for the Lie algebra $\widetilde{\mathfrak{g}}$, where we replace the Borel subalgebra $\widehat{\mathfrak{b}}$ of $\widehat{\mathfrak{g}}$ by the standard Borel subalgebra

$$\widetilde{\mathfrak{b}} := \mathfrak{g} \otimes \xi\mathbb{C}[\xi] \oplus \mathfrak{b} \otimes \xi^0 \oplus \mathbb{C}c.$$

Since

$$\begin{aligned} \widehat{\mathfrak{g}} &= \widehat{\mathfrak{b}} \oplus \widehat{\mathfrak{u}}_{\mathfrak{b}}^{-}, \\ \widetilde{\mathfrak{g}} &= \widetilde{\mathfrak{b}} \oplus \widehat{\mathfrak{u}}_{\mathfrak{b}}^{-}. \end{aligned}$$

It follows that any highest weight $\widehat{\mathfrak{g}}$-module is also a highest weight $\widetilde{\mathfrak{g}}$-module. Any quotient module of a highest weight module is also a highest weight module.

For $\widehat{\lambda} \in \widehat{\mathfrak{h}}^*$, let us now define the *Verma module*

$$\widehat{\mathcal{M}}(\widehat{\lambda}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{b}})} \mathbb{C}_{\widehat{\lambda}},$$

where $U(\widehat{\mathfrak{g}})$ is the universal enveloping algebra of $\widehat{\mathfrak{g}}$. $\mathbb{C}_{\widehat{\lambda}}$ is a 1-dimensional $\widehat{\mathfrak{b}}$-module on which $\widehat{\mathfrak{h}}$ acts by $\widehat{\lambda}$. Since $\mathbb{C}_{\widehat{\lambda}}$ is 1-dimensional it is killed by $[\mathfrak{b}, \mathfrak{b}]$. Further we know,

$$\widehat{\mathfrak{b}} = \widehat{\mathfrak{h}} \oplus [\widehat{\mathfrak{b}}, \widehat{\mathfrak{b}}].$$

The action of $U(\widehat{\mathfrak{b}})$ on $U(\widehat{\mathfrak{g}})$ is by multiplication on the right and the action of $U(\widehat{\mathfrak{g}})$ is by multiplication on the left. Now $\widehat{\mathcal{M}}(\widehat{\lambda})$ is a highest weight $\widehat{\mathfrak{g}}$-module. Any highest weight $\widehat{\mathfrak{g}}$-module is a quotient of $\widehat{\mathcal{M}}(\widehat{\lambda})$.

Similarly we can define $\widetilde{\mathcal{M}}(\widehat{\lambda})$, a Verma module for $\widetilde{\mathfrak{g}}$. Now we know that $\widetilde{\mathfrak{g}} \subset \widehat{\mathfrak{g}}$. Also $\widehat{\mathfrak{g}} = \widehat{\mathfrak{b}} \oplus \widehat{\mathfrak{u}}_{\mathfrak{b}}^{-}$ and $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{b}} \oplus \widehat{\mathfrak{u}}_{\mathfrak{b}}^{-}$.

By the Poincare-Birkoff-Witt theorem, we see $\widetilde{\mathcal{M}}(\widehat{\lambda})$ and $\widehat{\mathcal{M}}(\widehat{\lambda})$ are isomorphic. The isomorphism is induced by the canonical inclusion of $\widetilde{\mathfrak{g}} \subset \widehat{\mathfrak{g}}$. The $\widehat{\mathfrak{g}}$-module structure on $\widehat{\mathcal{M}}(\widehat{\lambda})$ is induced by the $\widetilde{\mathfrak{g}}$-module structure on $\widetilde{\mathcal{M}}(\widehat{\lambda})$.

Let $I_\ell(V)$ be a $\mathfrak{g}$-module such that c acts on V by ℓ and $P \otimes x$ acts on v by $P(0)x.v$. We now define the *generalized Verma module*

$$\widehat{\mathcal{M}}(V, \ell) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{p}}_+)} I_\ell(V),$$

where $\widehat{\mathfrak{p}}_+ = \mathfrak{g} \otimes \mathbb{C}[[\xi]] \oplus \mathbb{C}c$. We note that $\widehat{\mathfrak{p}}_+$ is nothing but the standard parabolic algebra defined earlier where $\mathfrak{p} = \mathfrak{g}$.

Similarly we can define the *generalized Verma module* of $\widetilde{\mathfrak{g}}$

$$\widetilde{\mathcal{M}}(V, \ell) = U(\widetilde{\mathfrak{g}}) \otimes_{U(\widetilde{\mathfrak{p}}_+)} I_\ell(V),$$

where $\widetilde{\mathfrak{p}}_+$ is defined by $\widetilde{\mathfrak{p}}_+ = \mathfrak{g} \otimes \mathbb{C}[\xi] \oplus \mathbb{C}c$. Define $\widehat{\mathfrak{u}}_{\mathfrak{g}}^{-} := \mathfrak{g} \otimes \xi^{-}\mathbb{C}[\xi^{-1}]$. So now

$$\begin{aligned} \widehat{\mathfrak{g}} &= \widehat{\mathfrak{p}}_+ \oplus \widehat{\mathfrak{u}}_{\mathfrak{g}}^{-}, \\ \widetilde{\mathfrak{g}} &= \widetilde{\mathfrak{p}}_+ \oplus \widehat{\mathfrak{u}}_{\mathfrak{g}}^{-}. \end{aligned}$$

Thus again using the last relation and Poincare-Birkoff-Witt theorem,

$$\widehat{\mathcal{M}}(V, \ell) = \widetilde{\mathcal{M}}(V, \ell).$$

We are interested in the case when V is generated by the highest weight vector v_+ such that $\mathfrak{h}$ acts on v_+ by $\lambda(h)$, where $\lambda \in \mathfrak{h}^*$. We define an element λ_ℓ of $\widehat{\mathfrak{h}}^*$ such that λ_ℓ restricted to $\mathfrak{h}$ is λ and $\lambda_\ell(c) = \ell$. We have a map,

$$\begin{array}{rcl} \pi : \widehat{\mathcal{M}}(\widehat{\lambda_\ell}) & \to & \widehat{\mathcal{M}}(V, \ell), \\ 1 \otimes_{\lambda_\ell} & \to & 1 \otimes v_+. \end{array}$$

Since v_+ generates $I_\ell(V)$ as a $\mathfrak{g}$-module, the map π is surjective. Hence we conclude that the $\widehat{\mathcal{M}}(V, \ell)$ is a highest weight $\widehat{\mathfrak{g}}$-module.

Next we claim that $\widehat{\mathcal{M}}(V, \ell)$ is a locally finite $\mathfrak{g}$-module.

Lemma 3. *$\widehat{\mathcal{M}}(V, \ell)$ is a locally finite $\mathfrak{g}$-module.*

Proof. We will use the Poincare-Birkoff-Witt theorem to construct an isomorphism and then later show that it is a $\mathfrak{g}$-module isomorphism. We know that $\widehat{\mathfrak{g}} = \widehat{\mathfrak{p}}_+ \oplus \widehat{\mathfrak{u}}_{\mathfrak{g}}^-$. Using Poincare-Birkoff-Witt theorem we get,

$$U(\widehat{\mathfrak{g}}) = U(\widehat{\mathfrak{p}}_+) \otimes_{\mathbb{C}} U(\widehat{\mathfrak{u}}_{\mathfrak{g}}^-).$$

We have produced an isomorphism of vector spaces,

$$i : U(\widehat{\mathfrak{u}}_{\mathfrak{g}}^-) \otimes_{\mathbb{C}} I_\ell(V) \longrightarrow U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{p}}_+)} I_\ell(V).$$

The action of $\mathfrak{g}$ on $U(\widehat{\mathfrak{u}}_{\mathfrak{g}}^-)$ is given by the adjoint action and the $\mathfrak{g}$ action on the target space is by its inclusion in $\widehat{\mathfrak{g}}$. If we show the above defined action makes the linear isomorphism a $\mathfrak{g}$-module isomorphism, we are done. This follows as $I_\ell(V)$ is a locally finite $\mathfrak{g}$-module and $U(\widehat{\mathfrak{u}}_{\mathfrak{g}}^-)$ is also a locally finite $\mathfrak{g}$-module under the adjoint action.

$$\begin{array}{rcl} i(x.(a \otimes v)) & = & i(adx(a) \otimes v + a \otimes x.v) \\ & = & i(adx(a) \otimes v) + i(a \otimes x.v) \\ & = & (adx)a \otimes v + a \otimes x.v \\ & = & (xa - ax) \otimes v + a \otimes x.v \\ & = & x \otimes (i(a \otimes v)). \end{array}$$

□

For $\lambda \in P_+$, let $V(\lambda)$ be the finite dimensional irreducible $\mathfrak{g}$ with highest weight λ. We define

$$\widehat{P} := \{\widehat{\lambda} \in \widehat{\mathfrak{h}}^* : \widehat{\lambda}_{|\mathfrak{h}} \in P_+ \text{ and } \widehat{\lambda}(c) - \widehat{\lambda}(\theta^\vee) \in \mathbb{Z}^+\}.$$

Let $\widehat{\lambda} \in \widehat{P}$ and $\ell = \widehat{\lambda}(c)$. In this chapter we fix ℓ. We define $\mathcal{M}_\lambda = \widehat{\mathcal{M}}(V(\lambda), \ell)$. Let us define a new $\widehat{\mathfrak{g}}$-module

$$\mathcal{H}_\lambda = \frac{\mathcal{M}_\lambda}{U(\widehat{\mathfrak{g}})((x_\theta \otimes \xi^{-1})^{\ell - \langle \theta, \lambda \rangle + 1} \otimes v_+)},$$

where x_θ is a nonzero element of $\mathfrak{g}_\theta$ and v_+ is a nonzero element of the line $\mathbb{C}v_+ \subset V(\lambda)$ which is stable under action of $\mathfrak{b}$. We also observe,

$$\widehat{\lambda}(\theta^\vee) = 2\frac{\langle \theta, \lambda \rangle}{\langle \theta, \theta \rangle}$$

and $\langle \theta, \theta \rangle = 2$. Thus $\widehat{\lambda}(\theta^\vee) = \langle \theta, \lambda \rangle$. Now by definition it is a quotient of a highest weight $\widehat{\mathfrak{g}}$-module, so it is also a highest weight $\widehat{\mathfrak{g}}$-module. We need to show, it is an integrable $\widehat{\mathfrak{g}}$-module. By the above lemma $\mathcal{H}_\lambda$ is a locally finite $\mathfrak{g}$-module. We still need to show, $\mathcal{H}_\lambda$ is a locally finite $\mathfrak{r}$-module. We state the following lemma without proof:

Lemma 4. *(a) Let $\mathfrak{s}$ be a Lie algebra and let $x \in \mathfrak{s}$ be any element. Define*

$$\mathfrak{s}_x := \{y \in \mathfrak{s} : (adx)^{n_y} y = 0 \textit{ for some } y \in \mathbb{N}\},$$

where $\mathbb{N}$ is the set of strictly positive integers. Then $\mathfrak{s}_x$ is a Lie subalgebra of $\mathfrak{s}$.

(b) For any representation (V, π) of $\mathfrak{s}$ and $x \in \mathfrak{s}$, we define $V_x = \{v \in V : \pi(x)^{n_v} v = 0$ for some $n_v \in \mathbb{N}\}$. Then V_x is a $\mathfrak{s}_x$-submodule of V.

(c) Let (V, π) be a representation of $\mathfrak{s}$ such that it is generated as a Lie algebra by the set

$$F_V = \{x \in \mathfrak{s} : adx \textit{ acting on } \mathfrak{s} \textit{ is locally finite and } \pi(x) \textit{ is locally finite}\}.$$

Then

(1) $\mathfrak{s}$ is spanned over $\mathbb{C}$ by F_V. In particular, if $\mathfrak{s}$ is spanned by the set F of its ad locally finite vectors, then F spans $\mathfrak{s}$.

(2) If $\dim \mathfrak{s} < \infty$, then any $v \in V$ lies in a finite dimensional $\mathfrak{s}$-submodule of V.

Thus we are now ready to prove the following theorem:

Theorem 4. *$\mathcal{H}_\lambda$ is a locally finite $\mathfrak{r}$-module.*

Proof. Let $x_\beta \in \mathfrak{g}_\beta$ be any root vector and $n \in \mathbb{Z}$, then we can show that

$$ad(x_\beta \otimes \xi^n) : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$$

is a locally nilpotent transformation. For $x = x_\theta \otimes \xi^{-1}$, $V = \mathcal{H}_\lambda$ and $\mathfrak{s} = \widetilde{\mathfrak{g}}$, apply the part (a) of lemma-4. We get $\mathfrak{s}_x$ is a subalgebra of $\mathfrak{s}$ and $\mathfrak{s}_x = \mathfrak{s}$. Clearly $1 \otimes v_+ \in V_x$. By part (b) of lemma-4, V_x is a submodule of V. Further the $\mathfrak{s}$-submodule of V generated by $1 \otimes v_+$ is whole of V. This follows from the fact $\widetilde{\mathcal{M}}(\widehat{\lambda})$ is isomorphic with $\widehat{\mathcal{M}}(\widehat{\lambda})$ and there is a surjective map π from $\widehat{\mathcal{M}}(\widehat{\lambda})$ to $\mathcal{H}_\lambda$. Thus $x_\theta \otimes \xi^{-1}$ acts locally nilpotently on the whole of V.

Using similar argument we can show that $x_{-\theta} \otimes \xi$ acts locally nilpotently on $\mathcal{H}_\lambda$. A $\mathfrak{sl}_2(\mathbb{C})$-module on which X and Y acts locally nilpotently is also a locally finite $\mathfrak{sl}_2(\mathbb{C})$-module. This follows from the part (c-1) of the lemma-4. We know that $\mathfrak{r}$ is the principal $\mathfrak{sl}_2(\mathbb{C})$ inside $\widehat{\mathfrak{g}}$. Thus $\mathcal{H}_\lambda$ is a locally finite as $\mathfrak{r}$-module. □

A $\mathbb{C}d \ltimes \widetilde{\mathfrak{g}}$-module V is said to be integrable if it is integrable as a $\widetilde{\mathfrak{g}}$-module. It is called a *highest weight* $\mathbb{C}d \ltimes \widetilde{\mathfrak{g}}$-module, if there exists a line $\mathbb{C}(v_+) \subset V$ which is stable under $\mathbb{C}d \ltimes \widetilde{\mathfrak{b}}$ and v_+ generates V as a $\mathbb{C}d \ltimes \widetilde{\mathfrak{g}}$-module. Similarly we can define a highest weight integrable $\mathbb{C}d \ltimes \widehat{\mathfrak{g}}$-module. We state the following theorem without proof. For a proof see [Kum2].

Theorem 5. *Any integrable highest weight $\mathbb{C}d \ltimes \widetilde{\mathfrak{g}}$-module is irreducible.*

We now state and prove a theorem that characterizes all highest weight integrable modules of the affine Lie algebra $\widehat{\mathfrak{g}}$.

Theorem 6. *Any nonzero integrable, highest weight $\widehat{\mathfrak{g}}$-module is isomorphic to $\mathcal{H}_\lambda$ with a unique $\widehat{\lambda} \in \widehat{P}$.*

Thus we have, $\widehat{\lambda}$ and $\mathcal{H}_\lambda$ are in a bijective correspondence with $\widehat{P}$ and the set of isomorphism classes of nonzero highest weight integrable $\widehat{\mathfrak{g}}$-modules. More $\mathcal{H}_\lambda$ is an irreducible $\widehat{\mathfrak{g}}$-module.

Proof. Let V be any non zero integrable highest weight $\widehat{\mathfrak{g}}$-module. $\mathbb{C}v_+$ be the line stable under $\widehat{\mathfrak{b}}$ and $\widehat{\lambda} \in \widehat{\mathfrak{h}}^*$ be the character by which $\widehat{\mathfrak{b}}$ acts on $\mathbb{C}v_+$. Since it is locally finite as $\mathfrak{g}$-module, we get that the submodule V^0 generated by the highest weight vector v_+ is finite dimensional. $\mathfrak{b}$ acts on v_+ by $\widehat{\lambda}_{|\mathfrak{h}}$ and it keeps the line $\mathbb{C}v_+$ stable. Using theorem 3 we get $\widehat{\lambda}_{|\mathfrak{h}} \in P_+$. From $\mathfrak{sl}_2(\mathbb{C})$ representation theory it follows that $\widehat{\lambda}(c) - \widehat{\lambda}(\theta^\vee) \in \mathbb{Z}^+$. Thus $\widehat{\lambda} \in \widehat{P}$.

Since $[\widehat{\mathfrak{b}}, \widehat{\mathfrak{b}}]$ annihilates v_+, we get that $\mathfrak{g} \otimes \xi\mathbb{C}[[\xi]]$ annihilates v_+. Hence it annihilates the whole of V^0. So we get a surjective map $\widehat{\mathfrak{g}}$-module map

$$\phi : \mathcal{M}_\lambda \to V.$$

It maps takes $I_\ell(V(\lambda)) \to V^0$ isomorphically as a $(\mathfrak{g} \oplus \mathbb{C}c)$-module, where $\lambda = \widehat{\lambda}_{|\mathfrak{h}}$.

Using $\mathfrak{sl}_2(\mathbb{C})$ representation theory and the isomorphism of $\mathfrak{r}$ with $\mathfrak{sl}_2(\mathbb{C})$, we get

$$(x_\theta \otimes \xi^{-1})^{\ell - \langle \theta, \lambda \rangle + 1} \otimes v_+ = 0,$$

where $\ell = \widehat{\lambda}(c)$. Thus the maps ϕ factors as a surjective map

$$\overline{\phi} : \mathcal{H}_\lambda \to V.$$

Now if we are able to show that $\mathcal{H}_\lambda$ is irreducible, we are done. The generalized Verma module has an action of d, where d acts on $U(\mathfrak{g} \otimes \xi^{-1}\mathbb{C}[\xi^{-1}]) \otimes_{\mathbb{C}} I_\ell(V)$ by derivation on $U(\mathfrak{g} \otimes \xi^{-1}\mathbb{C}[\xi^{-1}])$ and acts trivially on $I_\ell(V)$. This action descends down to a action on $\mathcal{H}_\lambda$. Now this is clearly an integrable highest weight $\mathbb{C}d \ltimes \widehat{\mathfrak{g}}$-module hence also a $\mathbb{C}d \ltimes \widetilde{\mathfrak{g}}$-module. Hence by using the previous theorem, we get $\mathcal{H}_\lambda$ is irreducible as a $\mathbb{C}d \ltimes \widetilde{\mathfrak{g}}$-module. So it is also as a $\mathbb{C}d \ltimes \widehat{\mathfrak{g}}$-module. We need to show that it is also irreducible as a $\widehat{\mathfrak{g}}$-module.

Let N be a $\widehat{\mathfrak{g}}$-submodule of $\mathcal{H}_\lambda$. We consider a decomposition of

$$\mathcal{H}_\lambda = \oplus_{i \in \mathbb{Z}^+} \mathcal{H}_\lambda(i),$$

where

$$\mathcal{H}_\lambda(i) := \{v \in \mathcal{H}_\lambda : d.v = -iv\}.$$

We now observe that for any $n \in \mathbb{Z}$ and $x \in \mathfrak{g}$,

$$(\xi^n \otimes x).\mathcal{H}_\lambda(i) \subset \mathcal{H}_\lambda(i-n).$$

For any nonzero vector $v \in \mathcal{H}_\lambda$, consider the decomposition of $v = \sum v_i$, where $v_i \in \mathcal{H}_\lambda(i)$. We set $|v| = \sum i$ such that $v_i \neq 0$. We choose a non zero vector $v^0 \in N$ such that $|v^0| \leq |v|$, for all nonzero vectors $v \in N$. Then we get $(\xi^n \otimes x).v^0 = 0$, for all $n \geq 1$ and $x \in \mathfrak{g}$. If not, then $|(\xi^n \otimes x).v^0| < |v^0|$, which is a contradiction to the choice of v^0.

Next we claim that $|v^0| = 0$. If $|v^0| > 0$, then take a nonzero component of $v^0_{i_0}$ with $i_0 > 0$. By the above we get that $(\xi^n \otimes x).v^0_{i_0} = 0$, for all $n \geq 1$ and $x \in \mathfrak{g}$. By using Poincare-Birkoff-Witt theorem, we see that the $\mathbb{C}d \ltimes \widehat{\mathfrak{g}}$-submodule generated by $v^0_{i_0}$ is a proper $\mathbb{C}d \ltimes \widehat{\mathfrak{g}}$-submodule of $\mathcal{H}_\lambda$ which is a contradiction to the irreducibility of $\mathcal{H}_\lambda$. Thus we get $|v^0| = 0$ and $v^0 \in \mathcal{H}_\lambda(0)$. Again, the Poincare-Birkoff-Witt theorem tell us that the $\mathbb{C}d \ltimes \widehat{\mathfrak{g}}$-submodule generated v^0 is same as $\widehat{\mathfrak{g}}$-submodule generated by v^0. Hence $N = \mathcal{H}_\lambda$. Thus $\mathcal{H}_\lambda$ is irreducible.

Define a $\mathfrak{g}$-submodule of $\mathcal{H}_\lambda$

$$\mathcal{H}^0_\lambda = \{v \in \mathcal{H}_\lambda : (\mathfrak{g} \otimes \xi\mathbb{C}[[\xi]]).v = 0\}.$$

Then clearly as a $\mathfrak{g}$-submodule of $\mathcal{H}_\lambda$, we have $1 \otimes V(\lambda) = \mathcal{H}_\lambda(0)^0$. We take the decomposition of

$$\mathcal{H}^0_\lambda = \bigoplus_{i \geq 0} \mathcal{H}_\lambda(i)^0.$$

Now for any $i > 0$, we claim that $\mathcal{H}_\lambda(i)^0 = 0$. If it is not, then the $\widehat{\mathfrak{g}}$-submodule of $\mathcal{H}_\lambda$ generated by $\mathcal{H}_\lambda(i)^0$ will be proper, which is a contradiction since $\mathcal{H}_\lambda$ is irreducible. Thus we get,

$$\mathcal{H}^0_\lambda = 1 \otimes V(\lambda).$$

Now if $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ are isomorphic, then the corresponding $\mathcal{H}^0_\lambda$ and $\mathcal{H}^0_\mu$ are also isomorphic. So we conclude that $V(\lambda)$ and $V(\mu)$ are isomorphic. From representation theory of finite dimensional complex simple lie algebras, we immediately get,

$$\widehat{\lambda}_{|\mathfrak{h}} = \widehat{\mu}_{|\mathfrak{h}}.$$

Since $\widehat{\lambda}$ and $\widehat{\mu}$ acts by the same scalar, we also get $\widehat{\lambda}(c) = \widehat{\mu}(c)$. Thus $\widehat{\lambda} = \widehat{\mu}$. This completes the proof of the theorem. □

2.3 Energy momentum tensor and filtration

Fix a positive integer ℓ. We define

$$P_\ell = \{\lambda \in P_+ : 0 \leq \langle \theta, \lambda \rangle \leq \ell\}.$$

The dominant weights λ are characterized by

$$w(\lambda) \leq \lambda,$$

for all elements w in the Weyl Group (W).

2.3.1 Energy momentum tensor

We use the following notation

$$X(n) = X \otimes \xi^n,$$
$$X(z) = \sum_{n\in\mathbb{Z}} X(n)z^{-n-1},$$

where z is a variable.

The *normal ordering* ${}^o_o \quad {}^o_o$ is defined as follows:

$$ {}^o_o X(n)Y(m) {}^o_o = \begin{cases} X(n)Y(m) & \text{if } n < m, \\ \frac{1}{2}(X(n)Y(m) + Y(m)X(n)) & \text{if } n = m, \\ Y(m)X(n) & \text{if } n > m. \end{cases}$$

We also have,

$$ {}^o_o X(n)X(m) {}^o_o = X(n)X(m) - n\delta_{n+m,o}\langle X, X\rangle.c.$$

We define the *Energy Momentum Tensor* to be

$$T(z) = \frac{1}{2(g^* + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} {}^o_o J^a(z)J^a(z) {}^o_o,$$

where J^a's form an orthonormal basis for the Lie algebra $\mathfrak{g}$ under the normalized Cartan Killing form $\langle\ ,\ \rangle$ and g^* is the *dual Coxeter number* (to be defined) and ℓ is the Level.

We define the n^{th} *Virasoro operator* as follows:

$$L_n = \frac{1}{2(g^* + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} \sum_{m\in\mathbb{Z}} {}^o_o J^a(m)J^a(n-m) {}^o_o.$$

If $n \neq 0$, the operator L_n acts on $\mathcal{H}_\lambda$ and by the last identity we do not need the normal ordering to define L_n. So it can be defined as

$$L_n = \frac{1}{2(g^* + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} \sum_{m\in\mathbb{Z}} J^a(m)J^a(n-m).$$

If $n = 0$, we need the normal ordering to define L_0. So we get,

$$L_0 = \frac{1}{2(g^* + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} \left(J^a J^a + \sum_{m=1}^{\infty} J^a(-m)J^a(m) \right).$$

We have the following :

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

L_0 is the generalization of the Casimir operator for the affine Lie algebra. We know that the Casimir element belongs to the center of the $U(\mathfrak{g})$. So on simple $\mathfrak{g}$-modules it acts by scalar multiplication. It is easy to compute the scalar for a irreducible highest weight $\mathfrak{g}$-module.

Lemma 5. *$\lambda \in P_+$ and $V(\lambda)$ be the corresponding irreducible highest weight $\mathfrak{g}$-module of highest weight λ. Then the Casimir element c acts by*

$$(\langle \lambda, \lambda \rangle + 2\langle \lambda, \rho \rangle).id,$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$.

Now $\mathfrak{g}$ acts on itself by adjoint. Under the adjoint representation, $\mathfrak{g}$ is irreducible. The highest weight of $\mathfrak{g}$ considered as an $ad(\mathfrak{g})$-module is θ. Define the dual Coxeter number

$$g^* = 1 + 2\langle \theta, \rho \rangle.$$

So the action of the Casimir element c on $\mathfrak{g}$ considered as a $ad(\mathfrak{g})$-module is given by the scalar

$$\langle \theta, \theta \rangle + 2\langle \theta, \rho \rangle = 2g^*.$$

We prove some very useful identities which we use for finding the commutation relation with an element of the affine Lie algebra and the Virasoro operator L_n.

Lemma 6. *For $X \in \mathfrak{g}$; $m, n \in \mathbb{Z}$,*

$$\sum_{a=1}^{\dim \mathfrak{g}} \{[X, J^a](m)J^a(n) + J^a(m)[X, J^a](n)\} = 0,$$

where J^a's form an orthonormal basis for the Lie algebra $\mathfrak{g}$ under the normalized Cartan Killing form.

Proof. We know that the Cartan killing form is ad-invariant, i.e.

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0,$$

for any element $X, Y, Z \in \mathfrak{g}$. Using the normalized Cartan form and writing $[X, J^a]$ in terms of the basis J^b's we get,

$$\begin{aligned}
\sum_{a=1}^{\dim \mathfrak{g}} [X, J^a](m)J^a(n) &= \sum_{a=1}^{\dim \mathfrak{g}} \sum_{b=1}^{\dim \mathfrak{g}} \{\langle [X, J^a], J^b \rangle J^b(m)J^a(n)\} \\
&= -\sum_{a=1}^{\dim \mathfrak{g}} \sum_{b=1}^{\dim \mathfrak{g}} \{\langle J^a, [X, J^b] \rangle J^b(m)J^a(n)\} \\
&= -\sum_{a=1}^{\dim \mathfrak{g}} \sum_{b=1}^{\dim \mathfrak{g}} J^b(m)\{\langle J^a, [X, J^b] \rangle J^a(n)\} \\
&= -\sum_{b=1}^{\dim \mathfrak{g}} \sum_{a=1}^{\dim \mathfrak{g}} J^b(m)\{\langle [X, J^b], J^a \rangle J^a(n)\} \\
&= -\sum_{b=1}^{\dim \mathfrak{g}} J^b(m)\ [X, J^b](n) \\
&= -\sum_{a=1}^{\dim \mathfrak{g}} J^a(m)[X, J^a](n).
\end{aligned}$$

This completes the proof. □

Proposition 2. *For $X \in \mathfrak{g}$ and $n, m \in \mathbb{Z}$, we have*

$$[L_n, X(m)] = -mX(n+m) \quad .$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c_\nu}{12}(n^3 - n)\delta_{n+m,0},$$

where $c_\nu = \frac{\ell \dim \mathfrak{g}}{g^* + \ell}$.

Thus the L_n's form a Virasoro algebra *with central charge*

$$c_\nu = \frac{\ell \dim \mathfrak{g}}{g^* + \ell}.$$

Proof. We prove the first statement, the proof of the second follows from similar computations as the proof of the first one.

$$\begin{aligned}
2(g^* + \ell)[X(m), L_n] &= \sum_{a=1}^{\dim \mathfrak{g}} \big([X(m), \sum_{j \in \mathbb{Z}} {}^o_o J^a(j) J^a(n-j){}^o_o]\big) \\
&= \sum_{a=1}^{\dim \mathfrak{g}} \big([X(m), \sum_{j \leq \frac{n}{2}} J^a(j) J^a(n-j)] + \big([X(m), \sum_{j > \frac{n}{2}} J^a(n-j) J^a(j)]\big) \\
&= \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j \leq \frac{n}{2}} [X(m), J^a(j) J^a(n-j)] + \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j > \frac{n}{2}} [X(m), J^a(n-j) J^a(j)].
\end{aligned}$$

Let us consider the first term on the right hand side of the last equation,

$$\begin{aligned}
&\sum_{a=1}^{\dim \mathfrak{g}} \sum_{j \leq \frac{n}{2}} [X(m), J^a(j) J^a(n-j)] \\
&= \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j \leq \frac{n}{2}} \big([X(m), J^a(j)] J^a(n-j) + J^a(j)[X(m), J^a(n-j)]\big) \\
&= \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j \leq \frac{n}{2}} \big([X, J^a](m+j) J^a(n-j) + J^a(j)[X, J^a](m+n-j)\big) \\
&\quad + \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j \leq \frac{n}{2}} \big(\ell m \langle X, J^a \rangle \delta_{m+j,0} J^a(n-j) + \ell m \langle X, J^a \rangle \delta_{m+n-j,0} J^a(j)\big).
\end{aligned}$$

The second term of the righthand side of the first equation in the proof looks like,

$$\begin{aligned}
&\sum_{a=1}^{\dim \mathfrak{g}} \sum_{j>\frac{n}{2}} [X(m), J^a(n-j)J^a(j)] \\
&= \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j>\frac{n}{2}} \big([X(m), J^a(n-j)]J^a(j) + J^a(n-j)[X(m), J^a(j)]\big) \\
&= \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j>\frac{n}{2}} \big([X, J^a](m+n-j)J^a(j) + J^a(n-j)[X, J^a](m+j)\big) \\
&\quad + \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j>\frac{n}{2}} \big(\ell m\langle X, J^a\rangle \delta_{m+n-j,0} J^a(j) + \ell m \langle X, J^a\rangle \delta_{m+j,0} J^a(n-j)\big).
\end{aligned}$$

Putting j by $n-j$ and reindexing, the first term on the right hand side of the last equation can be rewritten as,

$$\sum_{a=1}^{\dim \mathfrak{g}} \sum_{j<\frac{n}{2}} \big([X, J^a](m+j)J^a(n-j) + J^a(j)[X, J^a](m+n-j)\big).$$

Putting things together we get,

$$\begin{aligned}
&\sum_{a=1}^{\dim \mathfrak{g}} \sum_{j>\frac{n}{2}} [X(m), J^a(n-j)J^a(j)] \\
&= \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j<\frac{n}{2}} \big([X, J^a](m+j)J^a(n-j) + J^a(j)[X, J^a](m+n-j)\big) \\
&\quad + \sum_{a=1}^{\dim \mathfrak{g}} \sum_{j>\frac{n}{2}} \big(\ell m\langle X, J^a\rangle \delta_{m+n-j,0} J^a(j) + \ell m \langle X, J^a\rangle \delta_{m+j,0} J^a(n-j)\big).
\end{aligned}$$

Now,

$$\sum_{a=1}^{\dim \mathfrak{g}} \sum_{j\in\mathbb{Z}} \big(\ell m\langle X, J^a\rangle \delta_{m+j,0} J^a(n-j) + \ell m \langle X, J^a\rangle \delta_{m+n-j,0} J^a(j)\big) = 2m\ell X(m+n).$$

In the next computation we use the following facts:

$$\langle [X, J^a], J^a\rangle = 0, \text{ for all a}$$

and the Casimir element of the adjoint representation is given by

$$\sum_{a=1}^{\dim \mathfrak{g}} [[X, J^a], J^a] = 2g^* X.$$

So $2(g^*+\ell)[X(m),L_n]$ can be rewritten as,

$$\begin{aligned}
&2(g^*+\ell)[X(m),L_n]\\
&=2\sum_{a=1}^{\dim\mathfrak{g}}\sum_{j<\frac{n}{2}}\big([X,J^a](m+j)J^a(n-j)+J^a(j)[X,J^a](m+n-j)\big)\\
&\quad+\sum_{a=1}^{\dim\mathfrak{g}}([X,J^a](m+\frac{n}{2})J^a(\frac{n}{2})+J^a(\frac{n}{2})[X,J^a](m+\frac{n}{2})\\
&\quad+\sum_{a=1}^{\dim\mathfrak{g}}\sum_{j\in\mathbb{Z}}\big(\ell m\langle X,J^a\rangle\delta_{m+j,0}J^a(n-j)+\ell m\langle X,J^a\rangle\delta_{m+n-j,0}J^a(j)\big)\\
&=2\sum_{a=1}^{\dim\mathfrak{g}}\sum_{j<\frac{n}{2}}\big([X,J^a](m+j)J^a(n-j)+J^a(j)[X,J^a](m+n-j)\big)\\
&\quad+2\sum_{a=1}^{\dim\mathfrak{g}}J^a(\frac{n}{2})[X,J^a](m+\frac{n}{2})+\sum_{a=1}^{\dim\mathfrak{g}}[[X,J^a],J^a](m+n)\\
&=2\sum_{a=1}^{\dim\mathfrak{g}}\sum_{j<\frac{n}{2}}\big([X,J^a](m+j)J^a(n-j)+J^a(j)[X,J^a](m+n-j)\big)\\
&\quad+2\sum_{a=1}^{\dim\mathfrak{g}}J^a(\frac{n}{2})[X,J^a](m+\frac{n}{2})+2g^*X(m+n)\\
&\quad+\sum_{a=1}^{\dim\mathfrak{g}}\sum_{j\in\mathbb{Z}}\big(\ell m\langle X,J^a\rangle\delta_{m+j,0}J^a(n-j)+\ell m\langle X,J^a\rangle\delta_{m+n-j,0}J^a(j)\big).
\end{aligned}$$

Let $m\geq 0$, we show

$$\begin{aligned}
&\sum_{a=1}^{\dim\mathfrak{g}}\Big(\sum_{j<\frac{n}{2}}\big([X,J^a](m+j)J^a(n-j)+J^a(j)[X,J^a](m+n-j)\big)+J^a(\frac{n}{2})[X,J^a](m+\frac{n}{2})\Big)\\
&=(m-1)g^*X(m+n).
\end{aligned}$$

Using the lemma-6 we get,

$$\begin{aligned}
&\sum_{a=1}^{\dim\mathfrak{g}}\left(\sum_{j<\frac{n}{2}}\big([X,J^a](m+j)J^a(n-j)+J^a(j)[X,J^a](m+n-j)\big)+J^a(\frac{n}{2})[X,J^a](m+\frac{n}{2})\right)\\
&=\sum_{a=1}^{\dim\mathfrak{g}}\sum_{j<\frac{n}{2}}\{-J^a(m+j)[X,J^a](n-j)+J^a(j)[X,J^a](m+n-j)\}\\
&+\sum_{a=1}^{\dim\mathfrak{g}}J^a(\frac{n}{2})[X,J^a](m+\frac{n}{2}).
\end{aligned}$$

On the right hand side of the equation only the 1st term where $\frac{n}{2}-m<j<\frac{n}{2}$ survives, rest all gets killed. Thus we are left with

$$\sum_{a=1}^{\dim \mathfrak{g}} \sum_{\frac{n}{2}-m<j<\frac{n}{2}} -J^a(m+j)[X,J^a](n-j).$$

Let us analyze this.

$$\sum_{a=1}^{\dim \mathfrak{g}} \sum_{\frac{n}{2}-m<j<\frac{n}{2}} -J^a(m+j)[X,J^a](n-j) \;=\; \sum_{a=1}^{\dim \mathfrak{g}} \sum_{\frac{n}{2}-m<j<\frac{n}{2}} [X,J^a](m+j)J^a(n-j).$$

Reindexing j by $-j$, we get

$$\begin{aligned}
&\sum_{a=1}^{\dim \mathfrak{g}} \sum_{\frac{n}{2}-m<j<\frac{n}{2}} [X,J^a](m+j)J^a(n-j)\\
&= \sum_{a=1}^{\dim \mathfrak{g}} \sum_{m-\frac{n}{2}>j>-\frac{n}{2}} [X,J^a](m-j)J^a(n+j)\\
&= \sum_{a=1}^{\dim \mathfrak{g}} \sum_{m-\frac{n}{2}>j>-\frac{n}{2}} \{J^a(n+j)[X,J^a](m-j) + [[X,J^a],J^a](m+n)\}\\
&= \sum_{a=1}^{\dim \mathfrak{g}} \sum_{m-\frac{n}{2}>j>-\frac{n}{2}} \frac{1}{2}\{J^a(n+j)[X,J^a](m-j) + J^a(n+j)[X,J^a](m-j)\}\\
&\;+ \sum_{a=1}^{\dim \mathfrak{g}} (m-1)[[X,J^a],J^a](m+n)\\
&= \sum_{a=1}^{\dim \mathfrak{g}} \sum_{m-\frac{n}{2}>j>-\frac{n}{2}} \frac{1}{2}\{[X,J^a](m-j)J^a(n+j) + J^a(m+j)[X,J^a](m-j)\}\\
&\;+ \sum_{a=1}^{\dim \mathfrak{g}} (m-1)[[X,J^a],J^a](m+n) - \sum_{a=1}^{\dim \mathfrak{g}} \frac{(m-1)}{2}[[X,J^a],J^a](m+n)\\
&= \sum_{m-\frac{n}{2}>j>-\frac{n}{2}} \sum_{a=1}^{\dim \mathfrak{g}} \frac{1}{2}\{[X,J^a](m-j)J^a(n+j) + J^a(m+j)[X,J^a](m-j)\}\\
&\;+ \sum_{a=1}^{\dim \mathfrak{g}} \frac{(m-1)}{2}[[X,J^a],J^a](m+n).
\end{aligned}$$

Pairing the terms on the right hand side of the above equation in opposite order, i.e. pair the first one on the left with the last one on the right, the second one on the left with the second to last one on the right and applying lemma-6, it follows that

$$\sum_{m-\frac{n}{2}>j>-\frac{n}{2}} \sum_{a=1}^{\dim \mathfrak{g}} \frac{1}{2}\{[X,J^a](m-j)J^a(n+j) + J^a(m+j)[X,J^a](m-j)\} = 0.$$

Thus,

$$\begin{aligned}\sum_{a=1}^{\dim\mathfrak{g}}\sum_{\frac{n}{2}-m<j<\frac{n}{2}} -J^a(m+j)[X,J^a](n-j) &= \sum_{a=1}^{\dim\mathfrak{g}}\frac{(m-1)}{2}[[X,J^a],J^a](m+n)\\ &= \frac{(m-1)}{2}\sum_{a=1}^{\dim\mathfrak{g}}[J^a,[J^a,X]](m+n)\\ \text{(Using the lemma on Casimir operators)} &= (m-1)g^*X(m+n).\end{aligned}$$

This completes the proof. The second claim follows the same way with little more computations. □

Corollary 1.

$$[L_n, X(z)] = z^n(z\frac{d}{dz}+n+1)X(z).$$

For $l(z), f(z) \in \mathbb{C}((z))$, let

$$X[f] = \operatorname*{Res}_{z=0}(X(z)f(z)dz)$$

and

$$T[\underline{\ell}] = \operatorname*{Res}_{z=0}(T(z)\ell(z)dz)$$

where $\underline{\ell} = \ell(z)dz$. Under this notation we get,

$$\begin{aligned}T[\xi\frac{d}{d\xi}] &= \operatorname*{Res}_{\xi=0}(\sum_{n\in\mathbb{Z}} L_n\xi^{-n-2}\xi\frac{d}{d\xi})\\ &= \operatorname*{Res}_{\xi=0}(\sum_{n\in\mathbb{Z}} L_n\xi^{-n-1}\frac{d}{d\xi})\\ &= L_0.\end{aligned}$$

Thus, if we identify the derivation with $d = -L_0$, we get a L_0 action on $\mathcal{H}_\lambda$. $X[f]$ and $T[\underline{\ell}]$ also acts on $\mathcal{H}_\lambda$.

Proposition 3.

$$X[f] = X \otimes f(\xi).$$

$$[T[\underline{\ell}], X[f]] = -X[\underline{\ell}(f)].$$

$$[T[\underline{\ell}], T[\underline{m}]] = -T[[\underline{\ell}, \underline{m}]] + \frac{c_\nu}{12}\operatorname*{Res}_{z=0}(\ell'''mdz=0).$$

Proof. Let $f(z) = \sum_{i=-n_0}^{\infty} a_i z^i$. Then,

$$\begin{aligned}X[f] &= \operatorname*{Res}_{z=0}(X(z).\sum_{i=-n_0}^{\infty}a_iz^i)\\ &= \operatorname*{Res}_{z=0}(\sum_{m\in\mathbb{Z}}X(m)z^{-m-1}.\sum_{i=-n_0}^{\infty}a_iz^i)\\ &= \sum_{i=-n_0}^{\infty}a_iX(i)\\ &= X\otimes f(\xi).\end{aligned}$$

Let $\ell(z) = \sum_{i=-n_1}^{\infty} b_i z^i$. Then,

$$\begin{aligned}
[T[\underline{\ell}], X[f]] &= \sum_i [b_{i+1}L_i, X \otimes f(\xi)] \\
&= \sum_i b_{i+1}[L_i, X \otimes f(\xi)] \\
&= \sum_i \sum_m b_{i+1}a_m[L_i, X(m)] \\
&= \sum_i \sum_m b_{i+1}a_m(-mX(m+i)) \\
&= -\sum_i \sum_m m b_{i+1}a_m X(m+i) \\
&= -X[\underline{\ell}[f]].
\end{aligned}$$

Let $m(z) = \sum_{i=-n_2}^{\infty} c_i z^i$, where $\underline{m} = m\frac{d}{dz}$.

$$\begin{aligned}
[[T[\underline{\ell}], T[\underline{m}]] &= \sum_{k,j} [\ell_{j+1}L_j, m_{k+1}L_k] \\
&= \sum_{k,j} \ell_{j+1}m_{k+1}[L_j, L_k] \\
&= \sum_{k,j} \ell_{j+1}m_{k+1}\{(j-k)L_{j+k} + \frac{c_\nu}{12}(j^3 - j)\delta_{j+k,0}\}.
\end{aligned}$$

Now,

$$[\underline{\ell}, \underline{m}] = \sum_n \Big(\sum_{j+k=n} \ell_{j+1}m_{k+1}(k-j) \Big) z^{n+1}\frac{d}{dz} \text{ and}$$

$$\sum_{k,j} \ell_{j+1}m_{k+1}(j^3 - j)\delta_{j+k,0} = \sum_j \ell_{j+1}m_{-j+1}(j^3 - j) = \operatorname*{Res}_{z=0}(\ell'''(z)m(z)dz).$$

This completes the proof. □

2.3.2 Filtration on $\mathcal{H}_\lambda$

In this section we define a filtration $\{F_\bullet\}$ on the highest weight integrable $\widehat{\mathfrak{g}}$-module coming from the derivation. We define

$$\mathcal{H}_\lambda(i) = \{v \in \mathcal{H}_\lambda : L_0.v = (\Delta_\lambda + i).v\},$$

where

$$\Delta_\lambda = \frac{\langle \lambda, \lambda \rangle + 2\langle \lambda, \rho \rangle}{2(g^* + \ell)}.$$

Since for $m > 0$, $J^a(m)$ kills v_λ, we get

$$
\begin{aligned}
L_0.v_\lambda &= \frac{1}{2(g^*+\ell)}\sum_{a=1}^{\dim\mathfrak{g}}\left(J^aJ^a+\sum_{m=1}^{\infty}J^a(-m)J^a(m)\right).v_\lambda \\
&= \frac{1}{2(g^*+\ell)}\sum_{a=1}^{\dim\mathfrak{g}}J^aJ^a.v_\lambda \\
&= \frac{\langle\lambda,\lambda\rangle+2\langle\lambda,\rho\rangle}{2(g^*+\ell)}.v_\lambda \\
&= \Delta_\lambda v_\lambda.
\end{aligned}
$$

For $m > 0$ and $v \in \mathcal{H}_\lambda(0)$ we get,

Lemma 7.

$$L_0X(-m).v = (\Delta_\lambda + m)X(-m).v.$$

Proof.

$$
\begin{aligned}
L_0X(-m).v &= X(-m)L_0.v + [L_0, X(-m)].v \\
&= X(-m).\Delta_\lambda.v + m.X(-m).v \\
&= \Delta_\lambda X(-m).v + m.X(-m).v \\
&= (\Delta_\lambda + m)X(-m).v.
\end{aligned}
$$

□

Lemma 8. *For non negative integers $m_1, m_2, \cdots, m_k$, $v \in \mathcal{H}_\lambda(0)$ and $X_1, X_2, \cdots, X_k \in \mathfrak{g}$ we have,*

$$L_0X_1(-m_1)\cdots X_k(-m_k).v = (\Delta_\lambda + m_1 + m_2 + \cdots + m_k)X_1(-m_1)\cdots X_k(-m_k).v$$

Proof. The proof of the above is by induction on k. The base case follows from the previous lemma and for the induction step we assume,

$$L_0X_1(-m_1)\cdots X_i(-m_i).v = (\Delta_\lambda + m_1 + m_2 + \cdots + m_i)X_1(-m_1)\cdots X_i(-m_i).v.$$

Hence,

$$
\begin{aligned}
& L_0X_1(-m_1)\cdots X_{i+1}(-m_{i+1}).v \\
=\ & (X_1(-m_1)L_0 + [X_1(-m_1), L_0])X_2(-m_2)\cdots X_{i+1}(-m_{i+1}).v \\
=\ & X_1(-m_1)L_0X_2(-m_2)\cdots X_{i+1}(-m_{i+1}).v + m_1X_1(-m_1).X_2(-m_2)\cdots X_{i+1}(-m_{i+1}).v \\
=\ & (\Delta_\lambda + m_2 + \cdots + m_{i+1})X_1(-m_1)X_2(-m_2)\cdots X_{i+1}(-m_{i+1}).v \\
& +m_1X_1(-m_1)X_2(-m_2)\cdots X_{i+1}(-m_{i+1}).v \\
=\ & (\Delta_\lambda + m_1 + m_2 + \cdots + m_i)X_1(-m_1)\cdots X_{i+1}(-m_{i+1}).v.
\end{aligned}
$$

□

Clearly

$$\mathcal{H}_\lambda = \bigoplus_{i=0}^{\infty} \mathcal{H}_\lambda(i).$$

We define the filtration $\{F_p\mathcal{H}_\lambda\}$ on $\mathcal{H}_\lambda$ to be,

$$F_p\mathcal{H}_\lambda = \bigoplus_{i=1}^{p} \mathcal{H}_\lambda(i).$$

Define

$$\mathcal{H}_\lambda^\dagger = \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_\lambda, \mathbb{C}).$$

$\mathcal{H}_\lambda^\dagger$ naturally gets a right $\widehat{\mathfrak{g}}$-module structure. Also

$$\mathcal{H}_\lambda^\dagger = \prod_{i=0}^{\infty} \mathcal{H}_\lambda^\dagger(i).$$

We can define a filtration $\{F^p\mathcal{H}_\lambda\}$ on $\mathcal{H}_\lambda^\dagger$ by,

$$F^p\mathcal{H}_\lambda^\dagger = \prod_{i\geq p} \mathcal{H}_\lambda{}^\dagger(i).$$

The right module structure on $\mathcal{H}_\lambda^\dagger$ can be converted to a left $\widehat{\mathfrak{g}}$-module structure by defining the new action as

$$X(n).f = -f.X(-n),$$

where $f \in \mathcal{H}_\lambda^\dagger$ and $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$. The action of $X(n)$ and L_m on $\mathcal{H}_\lambda$ are follows:

$$X(m)\mathcal{H}_\lambda(i) \subset \mathcal{H}_\lambda(i-m).$$

$$L_m\mathcal{H}_\lambda(i) \subset \mathcal{H}_\lambda(i-m).$$

Define $\lambda^\dagger = -w(\lambda)$ where w is the longest element of the Weyl group. We prove the following which establishes relation between the left and right $\widehat{\mathfrak{g}}$-modules.

Proposition 4. *There is a pairing* $(\ |\ )$ *unique up to a constant multiple on* $\mathcal{H}_\lambda \times \mathcal{H}_{\lambda^\dagger}$ *such that*

$$(X(n)u, v) + (u, X(-n)v) = 0.$$

The pairing restricted to $\mathcal{H}_\lambda(i) \times \mathcal{H}_{\lambda^\dagger}(i')$ *is trivial if* $i \neq i'$.

Proof. We prove the existence of the pairing by induction on the filtration of $\mathcal{H}_\lambda$ and $\mathcal{H}_{\lambda^\dagger}$. For the case $p = 0$, we consider $V(\lambda) \otimes V(\lambda^\dagger)$. It contains a trivial 1-dimensional $\mathfrak{g}$-module generated by the element $0_\lambda \otimes 0_{\lambda^\dagger}$. On it, we have a bilinear form $(\ |\ )$ unique up to a constant multiple which is an element of $\mathrm{Hom}_{\mathfrak{g}}(V(\lambda) \otimes V(\lambda^\dagger), \mathbb{C})$.

Now assume that we have a bilinear form $(\ |\ ) \in \mathrm{Hom}_{\mathfrak{g}}(F_pH_\lambda \otimes F_pH_{\lambda^\dagger}, \mathbb{C})$. We want to define a form $(\ |\ ) \in \mathrm{Hom}_{\mathfrak{g}}(F_{p+1}\mathcal{H}_\lambda \otimes F_{p+1}\mathcal{H}_{\lambda^\dagger}, \mathbb{C})$.

Every element of $F_{p+1}\mathcal{H}_\lambda$ looks like $X(-m)u$ for some $u \in \mathcal{H}_\lambda$, $X \in \mathfrak{g}$ and $m \in \mathbb{Z}^+$ and $v \in F_{p+1}\mathcal{H}_{\lambda^\dagger}$. Define

$$(X(-m)u, v) = -(u, X(m)v).$$

Now $X(m)v \in F_{p+1-m}\mathcal{H}_\lambda$, so the right hand side is already defined. Thus the pairing makes sense. We have the property that

$$(X(-m)u, v) + (u, X(m)v) = 0.$$

This pairing $(\ |\ )$ satisfies the required properties. This completes the proof. □

Corollary 2. *There is a canonical left $\widehat{\mathfrak{g}}$-module isomorphism between*

$$\mathcal{H}_\lambda^\dagger \simeq \widehat{\mathcal{H}}_{\lambda^\dagger},$$

where $\widehat{\mathcal{H}}_{\lambda^\dagger}$ is the completion of the left $\widehat{\mathfrak{g}}$-module $\mathcal{H}_{\lambda^\dagger}$ under the filtration $\{F_\bullet\}$.

Chapter 3

Conformal Blocks and Propagation of Vacua

3.1 Conformal blocks

Let

$$\mathfrak{X} = (C; Q_1, Q_2, \cdots, Q_N; \eta_1, \eta_2 \cdots, \eta_N),$$

be the data associated to a N-pointed stable curve C as defined in section 1.1.2. In this section we want to define the space of covacua associated to a curve and the conformal blocks on $\mathfrak{X}$.

Let $\mathfrak{g}$ be a complex simple Lie algebra, fix a non negative integer ℓ and consider the set P_ℓ. For $\lambda_i \in P_\ell$ define

$$\vec{\lambda} = \{\lambda_1, \lambda_2, \cdots, \lambda_N\} \in P_\ell^N.$$

We define a new Lie algebra

$$\widehat{\mathfrak{g}}_N = \bigoplus_{j=1}^{N} \mathfrak{g} \otimes \mathbb{C}((\xi_j)) \oplus \mathbb{C}c,$$

where c belongs to the center of the Lie algebra $\widehat{\mathfrak{g}}$. The commutation relation which comes from the affine Lie algebra $\widehat{\mathfrak{g}}$ defined component wise i.e.

$$[X_j \otimes f_j, Y_j \otimes g_j] = [X_j, Y_j] \otimes f_j g_j + c.\langle X_j, Y_j \rangle \operatorname*{Res}_{\xi_j = 0} g_j df_j,$$

where $f_j, g_j \in \mathbb{C}((\xi_j))$ and $X_j, Y_j \in \mathfrak{g}$.

We have shown in lemma 2 of section 1.2 that $H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j))$ can be naturally embedded inside $\oplus_{j=1}^N \mathbb{C}((\xi_j))$ via the mapping t. Since the sum of the residues of a meromorphic 1-form on a Nodal curve is 0, we define another Lie subalgebra of $\widehat{\mathfrak{g}}_N$ by,

$$\widehat{\mathfrak{g}}(\mathfrak{X}) = \mathfrak{g} \otimes H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j)).$$

Let $\mathcal{H}_{\lambda_i}$ be the highest weight integrable $\widehat{\mathfrak{g}}_i$-module with highest weight λ_i, where

$$\widehat{\mathfrak{g}}_i = \mathfrak{g} \otimes \mathbb{C}((\xi_i)) \oplus \mathbb{C}.c.$$

For each $\vec{\lambda}$ we define,

$$\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \mathcal{H}_{\lambda_2} \otimes \cdots \otimes \mathcal{H}_{\lambda_N}.$$

Now $\mathcal{H}_{\vec{\lambda}}$ can be regarded as a $\widehat{\mathfrak{g}}_N$-module under the component wise action. The action on $\{X_i[f_i]\}$ of $\widehat{\mathfrak{g}}(\mathfrak{X})$ is given by,

$$\begin{aligned}
&(X_1 \otimes f_1, X_2 \otimes f_2, \cdots, X_N \otimes f_N).(v_1 \otimes v_2 \otimes \cdots \otimes v_N) \\
&= \sum_{j=1}^{N} \rho_j(X_j \otimes f_j).(v_1 \otimes v_2 \otimes \cdots \otimes v_N) \\
&= \sum_{j=1}^{N} v_1 \otimes \cdots \otimes ((X_j \otimes f_j).v_j) \otimes \cdots \otimes v_N,
\end{aligned}$$

where

$$X_j \otimes f_j \in \mathfrak{g} \otimes \mathbb{C}((\xi_j)),\ v_i \in \mathcal{H}_{\lambda_i}$$

and ρ_j is the representation which acts on the jth component.

Define

$$\mathcal{H}^{\dagger}_{\vec{\lambda}} = \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\lambda_1} \otimes \mathcal{H}_{\lambda_2} \otimes \cdots \otimes \mathcal{H}_{\lambda_N}, \mathbb{C}).$$

We have a natural $\widehat{\mathfrak{g}}$-invariant bilinear pairing,

$$\langle\ ,\ \rangle : \mathcal{H}^{\dagger}_{\lambda} \times \mathcal{H}_{\lambda} \longrightarrow \mathbb{C}.$$

$$\langle u, av\rangle = \langle ua, v\rangle,$$

where $u \in \mathcal{H}_{\lambda^{\dagger}}$, $a \in \widehat{\mathfrak{g}}$ and $v \in \mathcal{H}_{\lambda}$. We can extend this pairing to a $\widehat{\mathfrak{g}}_N$-invariant paring,

$$\langle\ ,\ \rangle : \mathcal{H}^{\dagger}_{\vec{\lambda}} \times \mathcal{H}_{\vec{\lambda}} \longrightarrow \mathbb{C},$$

defined by

$$\langle(h_1 \otimes h_2 \otimes \cdots \otimes h_N), (v_1 \otimes v_2 \otimes \cdots \otimes v_N)\rangle = \langle h_1, v_1\rangle\langle h_2, v_2\rangle \cdots \langle h_N, v_N\rangle,$$

where $v_i \in \mathcal{H}_{\lambda_i}$ and $h_i \in \mathcal{H}^{\dagger}_{\lambda_i}$. The pairing defined above is perfect. We now define the space of covacua associated to $\mathfrak{X}$.

The *space of covacua* associated to $\mathfrak{X}$ is defined as,

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) = \mathcal{H}_{\vec{\lambda}} / \widehat{\mathfrak{g}}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}$$

or in other words the orbits of the $\widehat{\mathfrak{g}}(\mathfrak{X})$ action on $\mathcal{H}_{\vec{\lambda}}$. We will prove that it is a vector space of finite dimension.

The *conformal block* or the *space of vacua* associated to $\mathfrak{X}$ is defined as,

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}), \mathbb{C}).$$

Remark: The space $H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j))$ can be embedded inside $\oplus_{j=1}^N \mathbb{C}((\xi_j))$ under the condition (5) of definition in section 1.1.2. We will show that condition (5) is not required as long as conformal blocks are concerned. This is because of the fact that we can add a point with trivial representation. Due to *propagation of vacua* (to be defined), the conformal block with an added point is canonically isomorphic to the conformal block without the added point. So while working with conformal blocks we can always assume that the condition (5) in the definition of stable N-pointed curves in section 1.1.2 is always satisfied.

The bilinear pairing $\langle \ , \ \rangle$ also induces an perfect pairing between

$$\langle \ , \ \rangle : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \times \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) \longrightarrow \mathbb{C}.$$

We observe $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ satisfies the following condition which we state as lemma,

Lemma 9.

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) = \{\Psi \in \mathcal{H}^\dagger_{\vec{\lambda}} : \Psi.\widehat{\mathfrak{g}}(\mathfrak{X}) = 0\}.$$

The condition in the above lemma is called the *gauge condition.* It plays a fundamental part in the theory of conformal blocks. We define the data $\mathfrak{X}$ for disconnected semistable curves $C = C_1 \bigsqcup C_2$. Let C_1 be a N_1 pointed semistable curves and $\mathfrak{X}_1$ be the data associated to it,

$$\mathfrak{X}_1 = (C_1; P_1, P_2, \cdots, P_{N_1}; s_1, s_2, \cdots, s_{N_1})$$

Similarly, let C_2 be a N_2 pointed semistable curves and $\mathfrak{X}_2$ be the data associated to it,

$$\mathfrak{X}_2 = (C_2; Q_1, Q_2, \cdots, Q_{N_2}; t_1, t_2, \cdots, t_{N_2})$$

Consider the curve $C = C_1 \bigsqcup C_2$. We define the data

$$\mathfrak{X} = \mathfrak{X}_1 \bigsqcup \mathfrak{X}_2$$

as,

$$\mathfrak{X} = (C; P_1, P_2, \cdots, P_{N_1}, Q_1, Q_2, \cdots, Q_{N_2}; s_1, s_2, \cdots, s_{N_1}, t_1, t_2, \cdots, t_{N_2}).$$

The data $\mathfrak{X}$ satisfies the conditions in the definition in section 1.1.2 except the connectivity of the curve. We can still define conformal block $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ and the space of covacua $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$. The following proposition follows directly from the definition.

Proposition 5. *Let $\vec{\lambda_1}$ lies in $P_\ell^{N_1}$ and $\vec{\lambda_2}$ lies in $P_\ell^{N_2}$. With the above settings there are canonical isomorphisms*

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}_1}(\mathfrak{X}_1) \otimes \mathcal{V}^\dagger_{\vec{\lambda}_2}(\mathfrak{X}_2),$$
$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) \simeq \mathcal{V}_{\vec{\lambda}_1}(\mathfrak{X}_1) \otimes \mathcal{V}_{\vec{\lambda}_2}(\mathfrak{X}_2);$$

where $\vec{\lambda} = (\vec{\lambda}_1, \vec{\lambda}_2)$.

We state and prove the following from [Sor].

Lemma 10. *Let $\mathcal{A}$ be a Lie algebra and $\mathcal{H}$ be a $\mathcal{A}$-module of finite type i.e. $\mathcal{H} = U(\mathcal{A}).V$, where V is a finite dimensional vector space. Suppose there exists a basis e_i of $\mathcal{A}$ such that the action of e_i on $\mathcal{H}$ is locally finite (i.e. the vector space generated by the action of the powers of e_i on $\mathcal{H}$ is finite dimensional). Let $\mathcal{A}_+ = \{X \in \mathcal{A} | X.V = 0\}$ and $\mathcal{K}$ be a Lie subalgebra of $\mathcal{A}$ such that $(\mathcal{K} + \mathcal{A}_+)$ has finite codimension in $\mathcal{A}$. Then $\mathcal{H}/\mathcal{K}\mathcal{H}$ is finite dimensional.*

Theorem 7. *The conformal block $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ and the space of covacua $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$ is finite dimensional.*

Proof. We apply the above lemma to our present case by choosing $\mathcal{K} = \widehat{\mathfrak{g}}(\mathfrak{X})$, $\mathcal{A} = \widehat{\mathfrak{g}}_N$ and $\mathcal{H} = \mathcal{H}_{\vec{\lambda}}$. Since we know that the representation $\mathcal{H}_{\vec{\lambda}}$ is locally finite, it satisfies the conditions of the previous lemma. The Riemann-Roch theorem tell us that $(\widehat{\mathfrak{g}}(\mathfrak{X}) + \widehat{\mathfrak{g}}_{N+})$ is of finite codimension in $\widehat{\mathfrak{g}}_N$. Let e_i be the elements such that

$$\mathcal{A} = (\mathcal{K} + \mathcal{A}_+) \oplus (\bigoplus_{j=1}^{N} \mathbb{C}e_j).$$

Thus we get,

$$U(\mathcal{A}) = \sum_{m_1, m_2, \cdots, m_N} U(\mathcal{K}) e_1^{m_1} \cdots e_N^{m_N} U(\mathcal{A}_+).$$

Now since $\mathcal{A}_+$ acts trivially on V, we get $U(\mathcal{A}_+)V = V$. Thus

$$\mathcal{H} = \sum_{m_1, m_2, \cdots, m_N} U(\mathcal{K}) e_1^{m_1} \cdots e_N^{m_N}.V.$$

Since e_i's act on V locally finitely, we get $\widetilde{L} = \sum_{m_1, m_2, \cdots, m_N} e_1^{m_1} \cdots e_N^{m_N}.V$ is finite dimensional. Thus $\mathcal{H} = U(\mathcal{K})\widetilde{L}$, where $\widetilde{L}$ is finite dimensional. Now there is a surjection from $\widetilde{L} \to \mathcal{H}/\mathcal{K}\mathcal{H}$. This completes the proof. □

3.2 Propagation of vacua

We start with the usual data

$$\mathfrak{X} = (C; Q_1, Q_2, \cdots, Q_N; \eta_1, \eta_2 \cdots, \eta_N).$$

We now add a new point and take the trivial representation corresponding to that point; so on the same curve C we have $N+1$ marked points. The new data is denoted by

$$\widetilde{\mathfrak{X}} = (C; Q_1, Q_2, \cdots, Q_N, Q_{N+1}; \eta_1, \eta_2 \cdots, \eta_N, \eta_{N+1}),$$

where $P = Q_{N+1}$ is the new added point and η_{N+1} is the formal neighborhood around the point P. The conformal block corresponding to $\widetilde{\mathfrak{X}}$ is denoted by $\mathcal{V}^\dagger_{\vec{\lambda},0}$. We have a natural mapping

$$\begin{aligned} \iota : \mathcal{H}_{\vec{\lambda}} &\to \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0, \\ v &\to v \otimes 0_P, \end{aligned}$$

where 0_P is the primitive element of the trivial representation at the point P. The map ι induces a surjection,

$$\widehat{\iota}: \mathcal{H}^{\dagger}_{\vec{\lambda}} \widehat{\otimes} \mathcal{H}^{\dagger}_{0} \to \mathcal{H}^{\dagger}_{\vec{\lambda}},$$

which when restricted descends to a map

$$\widehat{\iota}: \mathcal{V}^{\dagger}_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}) \to \mathcal{H}^{\dagger}_{\vec{\lambda}}.$$

We claim that the target space is actually $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$.

Lemma 11. *The mapping $\widehat{\iota}$ induced from the bilinear pairing is actually a map from*

$$\widehat{\iota}: \mathcal{V}^{\dagger}_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}) \to \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}).$$

Proof. Let $\widetilde{\Psi} = \widehat{\iota}(\Psi)$. We need to show that $\widehat{\iota}(\Psi)$ satisfies the gauge condition or in other words

$$\sum_{j=1}^{N} \widetilde{\Psi}.\rho_j(X[f]) = 0$$

on $\mathcal{H}_{\vec{\lambda}}$, for all $X \in \mathfrak{g}$ and for all $f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$. The proof follows easily from the gauge condition on Ψ. Choose a function $f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$. Since the function does not have any poles at the point P, we can consider f as an element of $H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N+1} Q_j))$. Now by the gauge condition on Ψ we know that for $u \otimes 0_P \in \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0$, we have

$$\sum_{j=1}^{N+1} \langle \Psi, \rho_j(X[f])(u \otimes 0) \rangle = 0.$$

Since the function is holomorphic at the point P and the representation corresponding to the point P is trivial; we get $\rho_{N+1}(X[f]).0_P = 0$. Thus for $u \otimes 0_P \in \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0$, we have

$$\sum_{j=1}^{N} \langle \Psi, \rho_j(X[f])(u \otimes 0_P) \rangle + \langle \Psi, \rho_{N+1}(X[f])0_P \rangle = 0.$$

Thus,

$$\sum_{j=1}^{N} \langle \Psi, \rho_j(X[f])(u \otimes 0_P) \rangle = 0.$$

So we get,

$$\sum_{j=1}^{N} \langle \widetilde{\Psi}, \rho_j(X[f])(u) \rangle = 0.$$

This completes the proof. □

Next we claim the map $\widehat{\iota}$ is injective.

Lemma 12.

$$\widehat{\iota}: \mathcal{V}^{\dagger}_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}) \longrightarrow \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$$

is injective.

Proof. With the notation same as that of the previous lemma we need to show that if $\widetilde{\Psi} = 0$ on $\mathcal{H}_{\vec{\lambda}}$, then Ψ is zero on $\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0$. We prove it by induction on the filtration $\{F_\bullet\}$ of $\mathcal{H}_0$. For the base case we know that

$$\langle \Psi, u \otimes 0_P \rangle = \langle \widetilde{\Psi}, u \rangle;$$

where $u \in \mathcal{H}_{\vec{\lambda}}$. But by assumption $\langle \widetilde{\Psi}, u \rangle = 0$. So the base case is done. Assume for $u \otimes v$ the lemma is true, where $u \in \mathcal{H}_{\vec{\lambda}}$ and $v \in F_p\mathcal{H}_0$. In other words,

$$\langle \Psi, u \otimes v \rangle = 0.$$

Now any element of $F_{p+1}\mathcal{H}_0$ can be written as $X(-m)v$; where m is a nonnegative integer and $v \in F_p\mathcal{H}_0$. We now choose a function $f \in H^0(C, \mathcal{O}_C(* \sum_{j=1}^{N+1}(Q_j)))$ such that at the point P we get,

$$f \equiv \frac{1}{\eta^m} \mod (\eta^M)$$

where M is a non negative integer such that for all $k \geq M$,

$$(X \otimes \eta^k).v = 0;$$

for $v \in F_p\mathcal{H}_0$. Such an M exists due to the local nilpotentcy for the action of the element X. The number M depends upon the vector v and the element X of the Lie algebra $\mathfrak{g}$. By gauge symmetry

$$\begin{aligned} \langle \Psi, u \otimes X(-m).v \rangle &= \langle \Psi, u \otimes X[f].v \rangle \\ &= -\sum_{j=1}^{N} \langle \Psi, \rho_j(X[f]).u \otimes v \rangle \\ &= 0 \quad \text{(By induction hypotheses)}. \end{aligned}$$

For any element $u \otimes v$ and $v \in \mathcal{H}_0$, we have proved

$$\langle \Psi, u \otimes v \rangle = 0.$$

Hence Ψ is zero on $\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0$. Thus the map $\widehat{\iota}$ is injective. This completes the proof. □

We recall some identity before we move on investigating further the map $\widehat{\iota}$. We know that for n even

$$\sum_{i=1}^{n-1} (-1)^{i-1} \binom{n}{i} = -2$$

and for n odd we get

$$\sum_{i=1}^{n-1} (-1)^{i-1} \binom{n}{i} = 0.$$

Gauge symmetry tell us,

$$\begin{aligned}
\langle \Psi, (X[f])^n.(u \otimes v) \rangle &= 0 \\
&= \sum_{j=0}^{n} \binom{n}{j} \langle \Psi, (X[f])^{n-j}.u \otimes (X[f])^j.v \rangle \\
&= \langle \Psi, (X[f])^n.u \otimes v \rangle + \langle \Psi, u \otimes (X[f])^n.v \rangle \\
&\quad + \sum_{j=1}^{n-1} \binom{n}{j} \langle \Psi, (X[f])^{n-j}.u \otimes (X[f])^j.v \rangle.
\end{aligned}$$

Similarly for $1 \leq k < n$, by gauge symmetry and similar techniques; we get

$$\begin{aligned}
\langle \Psi, (X[f])^{n-k}.(u \otimes (X[f])^k.v) \rangle &= 0 \\
&= \sum_{j=0}^{n-k} \binom{n-k}{j} \langle \Psi, (X[f])^{n-k-j}.u \otimes (X[f])^{k+j}.v \rangle.
\end{aligned}$$

We use another identity for $0 < k < n$ and $0 < k + j < n$:

$$\sum_{i=0}^{j} (-1)^i \binom{n}{k+i} \binom{n-k}{j-i} = \binom{n}{k+j}.$$

Now for n even, using the previous computation we get,

$$\begin{aligned}
&\sum_{i=1}^{n-1} \binom{n}{i} (-1)^{i+1} \langle \Psi, (X[f])^{n-i}.(u \otimes (X[f])^i.v) \rangle \\
&= \binom{n}{1} \langle \Psi, (X[f])^{n-1}.(u \otimes (X[f]).v) \rangle - \binom{n}{2} \langle \Psi, (X[f])^{n-2}.(u \otimes (X[f])^2.v) \rangle \\
&+ \binom{n}{3} \langle \Psi, (X[f])^{n-3}.(u \otimes (X[f])^3.v) \rangle - \cdots (-1)^n \binom{n}{n-1} \langle \Psi, X[f].(u \otimes (X[f])^{n-1}.v) \rangle \\
&= \sum_{j=1}^{n-1} \binom{n}{j} \langle \Psi, (X[f])^{n-j}.u \otimes (X[f])^j.v \rangle + 2 \langle \Psi, u \otimes (X[f])^n.v \rangle.
\end{aligned}$$

Similarly for n odd we get,

$$\begin{aligned}
&\sum_{i=1}^{n-1} \binom{n}{i} (-1)^{i+1} \langle \Psi, (X[f])^{n-i}.(u \otimes (X[f])^i.v) \rangle \\
&\qquad = \sum_{j=1}^{n-1} \binom{n}{j} \langle \Psi, (X[f])^{n-j}.u \otimes (X[f])^j.v \rangle.
\end{aligned}$$

Again by gauge symmetry;

$$\sum_{i=1}^{n-1} \binom{n}{i} (-1)^{i+1} \langle \Psi, (X[f])^{n-i}.(u \otimes (X[f])^i.v) \rangle = 0.$$

So we have,

$$\begin{aligned}\langle\Psi,(X[f])^n.(u\otimes v)\rangle &- \sum_{i=1}^{n-1}\binom{n}{i}(-1)^{i+1}\langle\Psi,(X[f])^{n-i}.(u\otimes(X[f])^i.v)\rangle\\ &= 0\\ &= -\langle\Psi,u\otimes(X[f])^n.v\rangle+\langle\Psi,(X[f])^n.u\otimes v\rangle \quad \text{if n is even,}\\ &= \langle\Psi,u\otimes(X[f])^n.v\rangle+\langle\Psi,(X[f])^n.u\otimes v\rangle \quad \text{if n is odd.}\end{aligned}$$

Let f be a function with a simple pole at the last marked point on a curve. We have proved the following lemma.

Lemma 13.

$$\begin{aligned}\langle\widetilde{\Psi},u\otimes(X_\theta[f])^n.v\rangle &= \langle\widetilde{\Psi},u\otimes(X_\theta(-1))^n.v\rangle\\ &= (-1)^n\sum_{\vec{n}}\left(\frac{n!}{n_1!n_2!\cdots n_N!}\right)\langle\prod_{j=1}^{N}\rho_j(X_\theta[f])^{n_j}.u\otimes v\rangle,\end{aligned}$$

where $\vec{n}$ is the collection of all N-tuples of nonnegative integers such that their sum is n.

In the next lemma we show that the mapping $\widehat{\iota}$ is also surjective. We prove it by explicitly constructing an element $\widetilde{\Psi}\in\mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}})$ corresponding to an element $\Psi\in\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$.

Lemma 14.

$$\widehat{\iota}:\mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}})\longrightarrow\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$$

is surjective.

Proof. Given an element $\Psi\in\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$, we construct an element $\widetilde{\Psi}\in\mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}})$. We do this by induction on the filtration $\{F_\bullet\}$ of the Verma module $\mathcal{M}_0$ corresponding to the trivial representation and forcing the gauge symmetry. We will later on show that the action of the element constructed descends to an action on $\mathcal{H}_{\vec{\lambda}}\otimes\mathcal{H}_0$.

For the base case, we want to define the linear functional $\widetilde{\Psi}$ as

$$\langle\widetilde{\Psi},u\otimes 0_P\rangle=\langle\Psi,u\rangle$$

The right hand side is well defined and for

$$f\in H^0(C,\mathcal{O}_C(*\sum_{j=1}^{N}Q_j))$$

we get,

$$\begin{aligned}\sum_{j=1}^{N}\langle\widetilde{\Psi},\rho_j(X[f]).u\otimes 0_P\rangle &= \sum_{j=1}^{N}\langle\Psi,\rho_j(X[f]).u\rangle\\ &= \Psi.X[f](u)\\ \text{(By gauge symmetry)} \quad &= 0.\end{aligned}$$

Assume that $\widetilde{\Psi}$ is defined for $\mathcal{H}_{\vec{\lambda}} \otimes F_p\mathcal{M}_0$ and with the condition that

$$\sum_{j=1}^{N} \langle \widetilde{\Psi}, \rho_j(X[f]).u \otimes v \rangle = 0;$$

for $u \in \mathcal{H}_{\vec{\lambda}}$, $v \in F_p\mathcal{M}_0$ and $f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$. We will now define $\widetilde{\Psi}$ on $\mathcal{H}_{\vec{\lambda}} \otimes F_{p+1}\mathcal{M}_0$. Any element of $F_{p+1}\mathcal{M}_0$ can be written as $X(-m)v$ for some nonnegative integer m and for some $v \in F_p\mathcal{M}_0$. We choose a function $f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N+1}(Q_j)))$ such that at the point P:

$$f \equiv \frac{1}{\eta^m} \qquad \mod (\eta^M),$$

where M is a non negative integer such that,

$$(X \otimes \eta^k).v = 0,$$

for $v \in F_p\mathcal{M}_0$. Such an M exists due to the local nilpotentcy of the action of the element X. Let us define

$$\langle \widetilde{\Psi}, u \otimes X(-m)v \rangle_f = -\sum_{j=1}^{N} \langle \widetilde{\Psi}, \rho_j(X[f])u \otimes v \rangle.$$

The right hand side is already defined, so the definition makes sense but we still need to check that it does not depend upon the choice of the function f.

Let g be another function satisfying the same properties as that of the function f. We consider the function $h = f - g$. This function does not have any poles at the point P and hence defines an element of $H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$. By induction we have,

$$\sum_{j=1}^{N} \langle \widetilde{\Psi}, \rho_j(X[h]).u \otimes v \rangle = 0.$$

Thus we get,

$$\langle \widetilde{\Psi}, u \otimes X(-m)v \rangle_f = \langle \widetilde{\Psi}, u \otimes X(-m)v \rangle_g$$

which implies that the definition is independent of the choice of the function f satisfying the required conditions.

We have to check for $g \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$;

$$\sum_{j=1}^{N} \langle \widetilde{\Psi}, \rho_j(X[g])u \otimes w \rangle = 0,$$

where $w \in F_{p+1}\mathcal{M}_0$ and $u \in \mathcal{H}_{\lambda}$. This follows from the definition since any w is of the form $w = X(-m)v$; where m and v are as before. Now,

$$\begin{aligned}
\sum_{j=1}^{N}\langle\widetilde{\Psi},\rho_j(X[g])u\otimes X(-m)v\rangle &= -\sum_{k=1}^{N}\sum_{j=1}^{N}\langle\widetilde{\Psi},\rho_k(X[f])\rho_j(X[g]).(u\otimes v)\rangle \\
&= -\sum_{j,k=1,\ j\neq k}^{N}\langle\widetilde{\Psi},\rho_k(X[f])\rho_j(X[g]).(u\otimes v)\rangle \\
&- \sum_{j=1}^{N}\langle\widetilde{\Psi},\rho_j(X[f])\rho_j(X[g]).(u\otimes v)\rangle \\
(\text{ By commutation relations }) &= -\sum_{j,k=1,\ j\neq k}^{N}\langle\widetilde{\Psi},\rho_j(X[g])\rho_k(X[f]).(u\otimes v)\rangle \\
&- \sum_{j=1}^{N}\langle\widetilde{\Psi},\rho_j(X[g])\rho_j(X[f]).(u\otimes v)\rangle \\
&= -\sum_{k=1}^{N}\sum_{j=1}^{N}\langle\widetilde{\Psi},\rho_j(X[g])\rho_k(X[f]).(u\otimes v)\rangle \\
&= -\sum_{k=1}^{N}\langle\widetilde{\Psi},X[g].(\rho_k(X[f]).(u\otimes v)\rangle \\
&= 0.
\end{aligned}$$

This proves the induction step. Also observe that the gauge symmetry is built into the definition. We can show that

$$\sum_{j=1}^{N+1}\widetilde{\Psi}.\rho_j(X[h]) = 0$$

on $\mathcal{H}_{\vec{\lambda}}\otimes F_{p+1}\mathcal{M}_0$, where $h\in H^0(C,\mathcal{O}_C(*\sum_{j=1}^{N+1}Q_j))$. This follows from the definition since,

$$\langle\widetilde{\Psi},u\otimes\rho_{N+1}(X[h])X(-m).v\rangle = -\sum_{j=1}^{N}\langle\widetilde{\Psi},\rho_j(X[h]).u\otimes X(-m)v\rangle.$$

Next we show that the element $\widetilde{\Psi}$ obeys the Lie algebra relations. In other words we need to show,

$$\begin{aligned}
&\langle\widetilde{\Psi},u\otimes X(-m_1)Y(-m_2)v\rangle - \langle\widetilde{\Psi},u\otimes Y(-m_2)X(-m_1)v\rangle \\
&\quad = \langle\widetilde{\Psi},u\otimes([X,Y](-m_1-m_2)+\ell.\langle X,Y\rangle.(-m_1)\delta_{m_1+m_2,0})v\rangle.
\end{aligned}$$

We choose elements

$$f,g\in H^0(C,\mathcal{O}_C(*\sum_{j=1}^{N+1}Q_j)),$$

such that it obeys the previous condition for $m=m_1$ and $m=m_2$ respectively. Thus by definition we get,

$$\begin{aligned}
&\langle\widetilde{\Psi}, u\otimes X(-m_1)Y(-m_2)v\rangle - \langle\widetilde{\Psi}, u\otimes Y(-m_2)X(-m_1)v\rangle\\
&= -\sum_{j=1}^{N}\langle\widetilde{\Psi}, \rho_j(X[f])u\otimes Y(-m_2)v\rangle\\
&\qquad\qquad + \sum_{i=1}^{N}\langle\widetilde{\Psi}, \rho_i(Y[g])u\otimes X(-m_1)v\rangle\\
&= \sum_{k=1}^{N}\sum_{j=1}^{N}\langle\widetilde{\Psi}, \rho_k(Y[g])\rho_j(X[f])u\otimes v\rangle\\
&\qquad\qquad - \sum_{k=1}^{N}\sum_{i=1}^{N}\langle\widetilde{\Psi}, \rho_k(X[f])\rho_j(Y[g])u\otimes v\rangle.
\end{aligned}$$

For $j \neq k$, the element $\rho_j(X[f])$ and $\rho_k(Y[f])$ commutes. Thus there is cancelation of similar terms. The only interesting case which remains is the case when $j = k$. Using commutation rules we get,

$$\begin{aligned}
&\langle\widetilde{\Psi}, u\otimes X(-m_1)Y(-m_2)v\rangle - \langle\widetilde{\Psi}, u\otimes Y(-m_2)X(-m_1)v\rangle\\
&\quad = \langle\widetilde{\Psi}, u\otimes ([X,Y](-m_1-m_2) + \ell.\langle X,Y\rangle.(-m_1)\delta_{m_1+m_2,0})v\rangle.
\end{aligned}$$

Thus we have shown that $\widetilde{\Psi}$ can be defined as a linear functional from $\mathcal{H}_{\vec{\lambda}}\otimes\mathcal{M}_0$ to $\mathbb{C}$.

It needs to be checked that $\widetilde{\Psi}$ descends to a linear functional on $\mathcal{H}_{\vec{\lambda}}\otimes\mathcal{H}_0$ where $u\in\mathcal{H}_{\vec{\lambda}}$ and θ is a long root of $\mathfrak{g}$. We need to prove the following:

$$\langle\widetilde{\Psi}, u\otimes (X_\theta(-1))^{\ell+1}.0_P\rangle = 0.$$

We first prove the above for large enough n. We know that $\widehat{\mathfrak{g}}$ acts locally nilpontently on the highest weight integrable representations. So, for $u\in\mathcal{H}_{\vec{\lambda}}$ there is an integer n depending on u such that for all $k\geq n/N$. We get

$$\rho_j(X_\theta[f])^k.0_P = 0,$$

where we choose f as in the previous proof with the condition that $m=1$. Thus by using lemma-13 for $v = 0_P$, we have

$$\begin{aligned}
&\langle\widetilde{\Psi}, u\otimes (X_\theta[f])^n.0\rangle = \langle\widetilde{\Psi}, u\otimes (X_\theta(-1))^n.0\rangle\\
= \quad & (-1)^n\sum_{\vec{n}}\left(\frac{n!}{n_1!n_2!\cdots n_N!}\right)\langle\prod_{j=1}^{N}\rho_j(X_\theta[f])^{n_j}.u\otimes 0\rangle\\
= \quad & 0,
\end{aligned}$$

where

$$\vec{n} = \{(n_1,n_2,\cdots,n_N) : \sum_{j=1}^{N} n_j = N\}.$$

Let us recall the definition of principal three dimensional Lie subalgebra $\widehat{\mathfrak{r}}$ of $\widehat{\mathfrak{g}}$.

$$\mathfrak{r} := \mathfrak{g}_\theta \otimes \xi^{-1} \oplus \mathfrak{g}_{-\theta} \otimes \xi \oplus \mathbb{C}(c - \theta^\vee),$$

where $\mathfrak{g}_\theta$ is the root space corresponding to the highest root θ and $\theta^\vee \in \mathfrak{h}$ is the coroot corresponding to θ.

We take $x_\theta \in \mathfrak{g}_\theta$ and $y_\theta \in \mathfrak{g}_{-\theta}$ satisfying $\langle x_\theta, y_\theta \rangle = 1$. This subalgebra $\mathfrak{r}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ under the identification, $X \to y_\theta \otimes \xi$, $Y \to x_\theta \otimes \xi^{-1}$ and $H \to c - \theta^\vee$.

Now since 0_P is the trivial representation of the Lie algebra $\mathfrak{g}$ and c acts on the representation by ℓ we have,

$$H.0_P = c.0_P = \ell 0_P.$$

Define a new vector subspace U of the Verma Module $\mathcal{M}_0$ by

$$U := \sum_{k=1}^{\infty} \mathbb{C} X_\theta(-1)^k.0_P.$$

By a simple computation, we see that the following holds

$$\begin{aligned} H.Y^k.0_P &= (\ell - 2k)Y^k.0_P, \\ XY^k.0_P &= (k\ell - k^2 + k)Y^{k-1}.0_P. \end{aligned}$$

From the above calculation, it follows that the vector space U is a $\mathfrak{r} = \mathfrak{sl}_2(\mathbb{C})$ module. We define

$$U_0 := \{v \in U : \langle \widetilde{\Psi}, u \otimes v \rangle = 0\},$$

for all $u \in \mathcal{H}_{\vec{\lambda}}$. By the identity we see that U_0 is also a $\mathfrak{r}$ module. For large enough integer m, we see that $Y^m.0_P \in U_0$. Thus the quotient space U/U_0 is a finite dimensional $\mathfrak{r}$ module. Let $\bar{0}_P$ be the image of 0_P in U/U_0. Then we have

$$H.\bar{0}_P = \ell \bar{0}_P.$$

Hence from $\mathfrak{sl}_2(\mathbb{C})$ representation theory,

$$Y^{\ell+1}.\bar{0}_P = 0_P.$$

This implies that

$$\langle \widetilde{\Psi}, u \otimes X_\theta(-1)^{\ell+1}.0_P \rangle = 0.$$

So $\widetilde{\Psi} \in \mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}})$ and $\widehat{\iota}(\widetilde{\Psi}) = \Psi$. This completes the proof. □

So combining the previous three lemmas we have proved the following important theorem often known as the *propagation of vacua.*

Theorem 8. *The canonical surjection $\widehat{\iota}$ is actually a canonical isomorphism between*

$$\mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}).$$

Corollary 3. *There is a canonical isomorphism between*

$$\mathcal{V}_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}) \simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}).$$

Chapter 4

Correlation Function and some global meromorphic forms on semistable curves

4.1 Correlation functions

Let C be a semistable curve and ω_C be the dualizing sheaf. Let C^M be the product of M copies of the curve.

$$C^M = \overbrace{C \times C \times \cdots \times C}^{M}. \tag{4.1}$$

Since C^M is locally a complete intersection and has singularities of codimension one, we can define the dualizing sheaf $\omega_{C^M}$of C^M and in this case

$$\omega_{C^M} = \omega_C^{\boxtimes M}.$$

$$\omega_C^{\boxtimes M} = \underbrace{\pi_1^*(\omega_C) \otimes \pi_2^*(\omega_C) \otimes \cdots \otimes \pi_M^*(\omega_C)}_{M},$$

where π_i is the projection to the ith factor from C^M to C.

We define the i, j-th diagonal

$$\Delta_{ij} = \{(P_1, P_2, \cdots, P_M) : P_i = P_j\}.$$

Fix a N-pointed semistable curve C and marked $Q_1, Q_2, \cdots, Q_N$. We have the following data:

$$\mathfrak{X} = (C; Q_1, Q_2, \cdots, Q_N; \eta_1, \eta_2, \cdots, \eta_N).$$

Let $P_1, P_2, \cdots, P_M, P_{M+1}$ be points on the curve C and define the following:

$$\widetilde{\mathfrak{X}} = (C; Q_1, Q_2, \cdots, Q_N, P_1, P_2, \cdots, P_M; \eta_1, \eta_2, \cdots, \eta_{N+M})$$

consisting of the points Q_i's and the first M of the P_i's. For all Q_i's and all points P_j's on the curve we have,

$$\widehat{\mathfrak{X}} = (C; Q_1, Q_2, \cdots, Q_N, P_1, P_2, \cdots, P_M, P_{M+1}; \eta_1, \eta_2, \cdots, \eta_{N+M}, \eta_{N+M+1})$$

Let

$$\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_N)$$

as before; where $\lambda_i \in P_\ell$. Similarly define

$$\vec{0}_M = \underbrace{(0, 0, \cdots, 0)}_{M};$$

where 0 is the weight associated to the trivial representation of the Lie algebra $\mathfrak{g}$ and

$$\vec{0}_{M+1} = \underbrace{(0, 0, \cdots, 0)}_{M+1}.$$

By the propagation of vacua consider the following isomorphism:

$$\widehat{\iota}_M : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}, \vec{0}_M}(\mathfrak{X}).$$

Similarly consider the isomorphism

$$\widehat{\iota}_1 : \mathcal{V}^\dagger_{\vec{\lambda}, \vec{0}_M}(\mathfrak{X}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}, \vec{0}_{M+1}}(\mathfrak{X}).$$

Using the above maps define the element $\widetilde{\Psi} = \widehat{\iota}_M(\Psi)$ and also define $\widehat{\Psi} = \widehat{\iota}_1(\widetilde{\Psi})$. Let z be a variable. The one form given by $\langle \widetilde{\Psi}, X(z)\widetilde{\Phi} \rangle dz$; has a local expansion around a point Q_j. For $\widetilde{\Psi} \in \mathcal{V}^\dagger_{\vec{\lambda}, \vec{0}_M}(\mathfrak{X})$ and $\widetilde{\Phi} \in \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_{\vec{0}_M}$ the local expansion about a point Q_j is given by,

$$\sum_{n \in \mathbb{Z}} \langle \widetilde{\Psi}, \rho_j(X(n))\widetilde{\Phi} \rangle \xi_j^{n+1} d\xi_j.$$

For each j define

$$\omega_j = \sum_{n \in \mathbb{Z}} \langle \widetilde{\Psi}, \rho_j(X(n))\widetilde{\Phi} \rangle \xi_j^{n+1} d\xi_j.$$

Locally ω_j defines a meromorphic form on the curve C around the point Q_j. We first proceed with a lemma that defines an element of the cotangent space to the curve C at a point P with parameter η.

Lemma 15. *For $\widetilde{u} \in \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_{\vec{0}_M}$ and $X \in \mathfrak{g}$*

$$\langle \widehat{\Psi}, \widetilde{u} \otimes X(-1)0_P \rangle d\eta$$

defines an element of $T^*_P C$ and is independent of the choice of the local parameter η.

Proof. We know that $X(n)$ acts locally nilpotently on $\mathcal{H}_{\vec{\lambda}}$. Let n_j be integers such that $\rho_j(X \otimes {\xi_j}^{n_j}).\widetilde{u} = 0$. Consider a function $f \in H^0(*(P + Q_1))$ such that

$$\begin{aligned} f &= \frac{1}{\eta} + \text{ regular at P and} \\ f &= 0 \mod ({\xi_j}^{n_j}) \text{ at } Q_j \text{ for all } j \neq i, \end{aligned}$$

where $\xi_j = {\eta_j}^{-1}(\xi)$.

We have by gauge symmetry,

$$\begin{aligned}\widehat{\Psi}.X[f] &= 0.\\ \widehat{\Psi}.X[f](\widetilde{u}\otimes 0_P) &= 0.\end{aligned}$$

Thus, we get

$$\begin{aligned}\sum_{j=1}^{N+M}\Big(\langle\widehat{\Psi},\rho_j(X[f])\widetilde{u}\otimes 0_P\rangle\Big) &+ \langle\widehat{\Psi},\widetilde{u}\otimes\rho_P(X[f])0_P\rangle = 0.\\ \text{Then,}\quad \langle\widehat{\Psi},\widetilde{u}\otimes\rho_P(X[f])0_P\rangle &= \langle\widehat{\Psi},\widetilde{u}\otimes 0_P\rangle\\ &= -\sum_{j=1}^{N+M}\langle\widehat{\Psi},\rho_j(X[f])\widetilde{u}\otimes 0_P\rangle\\ &= -\langle\widehat{\Psi},\rho_1(X[f])\widetilde{u}\otimes 0_P\rangle.\end{aligned}$$

If there is a holomorphic change of coordinates, or in other words if

$$\widetilde{\eta} = \sum_{i=1}^{\infty} a_i\eta^i,$$

where $a_1 \neq 0$. Then

$$\langle\widehat{\Psi},\widetilde{u}\otimes X(-1)0_P\rangle_{\widetilde{\eta}} = \frac{1}{a_1}\langle\widehat{\Psi},\widetilde{u}\otimes X(-1).0_P\rangle_{\eta}$$

which depends only on the first order deformation of the coordinates. Thus we see that $\langle\widehat{\Psi},\widetilde{u}\otimes X(-1)0_P\rangle d\eta$ defines an element of T_P^*C. Further $d\widetilde{\eta} = a_1 d\eta$, we get

$$\langle\widehat{\Psi},\widetilde{u}\otimes X(-1)0_P\rangle_{\widetilde{\eta}} d\widetilde{\eta} = \langle\widehat{\Psi},\widetilde{u}\otimes X(-1)0_P\rangle_{\eta} d\eta.$$

Hence it is independent of the choice of the local holomorphic coordinates. □

Lemma 16. *The ω_j's defined earlier comes form a global holomorphic form*

$$\omega \in H^0(C,\omega_C(*\sum_{i=1}^{N+M} Q_j))$$

such that

$$t(\omega) = (\omega_1,\omega_2,\cdots,\omega_{N+M}),$$

where the mapping $t = \oplus_{j=1}^{N+M} t_j$ and t_j is defined as the Laurent expansion of ω around the point Q_j.

Proof. If we are able to show that for any function $f \in H^0(C,\mathcal{O}_C(*\sum_{j=1}^{N+M} Q_j))$, the residue pairing defined earlier is zero; then we are done since $H^0(C,\mathcal{O}_C(*\sum_{j=1}^{N+M} Q_j))$ and $H^0(C,\omega_C(*\sum_{i=1}^{N+M} Q_j))$ are annihilators of each other.

Let $t_j(f) = f_j$ and $f_j(\xi_j) = \sum_{n=-n_0}^{\infty} a_{j,n}{\xi_j}^n$. We need to show that

$$\sum_{j=1}^{N+M} \operatorname*{Res}_{\xi_j=0} f_j(\xi_j)\omega_j d\xi_j = 0.$$

But

$$\begin{aligned}\sum_{j=1}^{N+M} \operatorname*{Res}_{\xi_j=0} f_j(\xi_j)\omega_j d\xi_j &= \sum_{j=1}^{N+M}\sum_{n\in\mathbb{Z}}\langle\widetilde{\Psi}, \rho_j(X(n))\widetilde{\Phi}\rangle a_{j,n} \\ &= \langle\widetilde{\Psi}, X\otimes t(f).\widetilde{\Phi}\rangle \\ &= 0.\end{aligned}$$

Since by gauge symmetry, we get $\widetilde{\Psi}.(X\otimes t(f)) = 0$. Thus there exists $\omega \in H^0(C, \omega_C(*\sum_{i=1}^{N+M} Q_j))$ such that $t(\omega) = (\omega_1, \omega_2, \cdots, \omega_{N+M})$. This completes the proof. □

Next we try to relate the form ω as cotangent vector to a point P on the curve with the cotangent vector in the first lemma given by $\langle\widehat{\Psi}, \widetilde{u}\otimes X(-1)0_P\rangle d\eta$.

Lemma 17. *As cotangent vectors to a point P on the curve C, we get ω and $\langle\widehat{\Psi}, \widetilde{u}\otimes X(-1)0_P\rangle d\eta$ coincide.*

Proof. Since ω is holomorphic at the point P, the value of ω at the point can be computed by $\mathrm{Res}_P \frac{1}{\eta}\omega$, where η is the parameter at the point P.

We know that $X(n)$ acts locally nilpotently on $\mathcal{H}_{\vec{\lambda}}$. Let n_j be integers such that $\rho_j(X\otimes {\xi_j}^{n_j}).\widetilde{u} = 0$ and $f\omega$ be holomorphic at the point Q_j, for $j \neq i$. Consider a function $f \in H^0(*(P+Q_i))$ on the curve C such that,

$$\begin{aligned} f &= \frac{1}{\eta} + \text{ regular at P and} \\ f &= 0 \mod (\xi_j^{n_j}) \text{ at } Q_j, \text{ for all } j\neq i \text{ where } \xi_j = \eta^{-1}(\xi).\end{aligned}$$

We have by gauge symmetry,

$$\widehat{\Psi}.X[f] = 0;$$

so we get

$$\widehat{\Psi}.X[f](\widetilde{u}\otimes 0_P) = 0.$$

Thus we have

$$\sum_{j=1}^{N+M}\Big(\langle\widehat{\Psi}, \rho_j(X[f])\widetilde{u}\otimes 0_P\rangle\Big) + \langle\widehat{\Psi}, \widetilde{u}\otimes\rho_P(X[f])0_P\rangle = 0.$$

$$\begin{aligned}\text{Hence,}\quad \langle\widehat{\Psi}, \widetilde{u}\otimes\rho_P(X[f])0_P\rangle &= \langle\widehat{\Psi}, \widetilde{u}\otimes X(-1).0_P\rangle \\ &= -\sum_{j=1}^{N+M}\langle\widehat{\Psi}, \rho_j(X[f])\widetilde{u}\otimes 0_P\rangle \\ &= -\langle\widehat{\Psi}, \rho_i(X[f])\widetilde{u}\otimes 0_P\rangle \\ &= -\langle\widetilde{\Psi}, \rho_i(X[f])\widetilde{u}\rangle.\end{aligned}$$

We also have,

$$
\begin{aligned}
\operatorname*{Res}_P(\frac{1}{\eta}\omega) &= \operatorname*{Res}_P(f\omega) \\
&= -\sum_{j=1}^{N+M} \operatorname*{Res}_{Q_j}(f\omega) \\
&= -\operatorname*{Res}_{Q_i}(f\omega) \\
&= -\operatorname*{Res}_{\xi_i=0}\left(f_i(\xi_i)\sum_{n\in\mathbb{Z}}\langle\widetilde{\Psi},\rho_i(X(n))\widetilde{u}\rangle{\xi_i}^{-n-1}d\xi_i\right) \\
&= -\langle\widetilde{\Psi},\rho_i(X[f])\widetilde{u}\rangle \\
&= -\langle\widetilde{\Psi},\rho_i(X[f])\widetilde{u}\rangle \\
&= \langle\widehat{\Psi},\widetilde{u}\otimes X(-1).0_P\rangle.
\end{aligned}
$$

□

Let $\widetilde{u} = u\otimes X_1(-1)0_{P_1}\otimes X_2(-1)0_{P_2}\otimes\cdots\otimes X_M(-1)0_{P_M}$ where $X_i's\in\mathfrak{g}$ and P_i's are points on the curve. Then our above discussion shows that for all i,j,k,m; $P_i\neq Q_j$ and $P_k\neq P_m$; the element

$$\langle\widetilde{\Psi},\widetilde{u}\rangle d\eta_1 d\eta_2\cdots d\eta_M$$

defines an element of $T^*_{P_1}C\otimes T^*_{P_2}C\otimes\cdots\otimes T^*_{P_M}C$.

Since the points where that form has poles has codimension greater than equal to 2. By Hartog's theorem, we conclude it is an element of

$$H^0(C^M,\omega_C^{\boxtimes M}(\sum_{i<j}*\Delta_{ij}+\sum_{i=1}^{M}\sum_{j=1}^{N}*\pi_i^{-1}(Q_j))).$$

A holomorphic section of the above sheaf is denoted by,

$$\langle\Psi,X_1(P_1)X_2(P_2)\cdots X_M(P_M)u\rangle dP_1dP_2\cdots dP_M.$$

Combining the three lemma's and the above discussions we have proved the following theorem:

Theorem 9. *Let $\Psi\in\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$; $X_1,X_2,\cdots,X_M\in\mathfrak{g}$ and $\Phi\in\mathcal{H}_{\vec{\lambda}}$. Define the element*

$$F=\langle\Psi,X_1(P_1)X_2(P_2)\cdots X_M(P_M)\Phi\rangle dP_1dP_2\cdots dP_M$$

of

$$H^0(C^M,\omega_C^{\boxtimes M}(\sum_{i<j}*\Delta_{ij}+\sum_{i=1}^{M}\sum_{j=1}^{N}*\pi_i^{-1}(Q_j))),$$

where

$$\Delta_{ij}=\{(P_1,P_2,\cdots,P_M):P_i=P_j\}.$$

The form F has the following properties:

(1) when $M = 0$ then it is the canonical pairing,

(2) F is linear in Φ and multi-linear in X_i,

(3) For $\sigma \in S_M$ the form F is independent under the action of σ; where S_M is the permutation group on M letters.

For $X \in \mathfrak{g}$ and $\xi_k = \eta_k^{-1}(\xi)$, let us consider

$$\langle \Psi, X(\xi_k) X_1(P_1) X_2(P_2) \cdots X_M(P_M).\Phi \rangle d\xi_k.$$

Using lemma-16, it can be expressed in the form of

$$\langle \Psi, X(\xi_k).X_1(-1)0_{P_1} \otimes X_2(-1)0_{P_2} \otimes \cdots \otimes X_M(-1)0_{P_M}.\Phi \rangle d\xi_k.$$

It can be rewritten as,

$$\begin{aligned}
&\langle \Psi, X(\xi_k).X_1(-1)0_{P_1} \otimes X_2(-1)0_{P_2} \otimes \cdots \otimes X_M(-1)0_{P_M} \otimes \Phi \rangle d\xi_k \\
&= \sum_{n \in \mathbb{Z}} \langle \Psi, X_1(-1)0_{P_1} \otimes X_2(-1)0_{P_2} \otimes \cdots \otimes X_M(-1)0_{P_M} \rho_k(X(n))\Phi \rangle {\xi_k}^{-n-1} d\xi_k \\
&= \sum_{n \in \mathbb{Z}} \langle \Psi, X_1(P_1) X_2(P_2) \cdots X_M(P_M).\rho_k(X(n))\Phi \rangle {\xi_k}^{-n-1} d\xi_k.
\end{aligned}$$

So from the above computation, we get the following theorem:

Theorem 10.

$$\begin{aligned}
\operatorname*{Res}_{\xi_k = 0} &\left(\xi_k^n \langle \Psi, X(\xi_k) X_1(P_1) X_2(P_2) \cdots X_M(P_M).\Phi \rangle \right) d\xi_k \\
&= \langle \Psi, X_1(P_1) X_2(P_2) \cdots X_M(P_M).(\rho_k(X(n)).\Phi) \rangle.
\end{aligned}$$

Using the above theorem we can get a local meromorphic expansion

$$\begin{aligned}
&\langle \Psi, X(\xi_k) X_1(P_1) X_2(P_2) \cdots X_M(P_M).\Phi \rangle d\xi_k \\
&\quad = \sum_{n \in \mathbb{Z}} \langle \Psi, X_1(P_1) X_2(P_2) \cdots X_M(P_M).(\rho_k(X(n)).\Phi) \rangle {\xi_k}^{-n-1} d\xi_k.
\end{aligned}$$

Let P be a point on the curve C and U be a coordinate neighborhood of P with coordinate z centered at P. P' be another point in U close to P and $w = z - z(P')$ be a coordinate of the point P'. Consider the function

$$f \in H^0(C, \mathcal{O}_C(*(P + \sum_{i=1}^{N+M} Q_j)))$$

such that

$$f = \frac{1}{z} + \text{regular at P}.$$

We can adjust the coordinate z such that $f = \frac{1}{z}$ at the point P. Also let $a = z(P')$, so the expansion of f around the point P' looks like

$$\begin{aligned} f &= \frac{1}{z-a+a} \\ &= \frac{1}{w+a} \\ &= \frac{1}{a(1+\frac{w}{a})} \\ &= \frac{1}{a}(1-\frac{w}{a}+\frac{w^2}{a^2}-\cdots). \end{aligned}$$

Consider the following element of $T^*_P(C) \otimes T^*_{P'}(C)$

$$\langle \widehat{\Psi}^*, X(-1)0_P \otimes Y(-1)0_{P'} \otimes \widetilde{\Phi} \rangle dzdw,$$

where

$$\widetilde{\Phi} = X_1(-1)0_{P_1} \otimes X_2(-1)0_{P_2} \otimes \cdots \otimes X_M(-1)0_{P_M} \otimes \Phi$$

and $\widehat{\Psi}^*$ is the image of $\widehat{\Psi}$ under the propagation of vacua. By gauge symmetry, we get

$$\widehat{\Psi}^*.X[f] = 0.$$

which implies

$$\widehat{\Psi}^* X[f].(0_P \otimes Y(-1)0_{P'} \otimes \widetilde{\Phi}) = 0.$$

Now,

$$\begin{aligned} &\langle \widehat{\Psi}^*, \rho_1(X[f])0_P \otimes Y(-1)0_{P'} \otimes \widetilde{\Phi} \rangle \\ &\quad = \langle \widehat{\Psi}^*, X(-1)0_P \otimes Y(-1)0_{P'} \otimes \widetilde{\Phi} \rangle \\ &\quad = -\sum_{j=1}^{N+M} \Big(\langle \widehat{\Psi}, Y(-1)0_{P'} \otimes \rho_j(X[f])\widetilde{\Phi} \rangle \Big) + \langle \widehat{\Psi}, \rho_{P'}(X[f])Y(-1)0_{P'} \otimes \widetilde{\Phi} \rangle. \end{aligned}$$

The first term on the right hand side in the above expression

$$\sum_{j=1}^{N+M} \langle \widehat{\Psi}, Y(-1)0_{P'} \otimes \rho_j(X[f])\widetilde{\Phi} \rangle$$

can be rewritten as

$$\sum_{j=1}^{N+M} \langle \widehat{\Psi}, Y(P').\rho_j(X[f])\widetilde{\Phi} \rangle;$$

which is holomorphic at the point P'. The second term can be simplified and expanded in the following way:

$$
\begin{aligned}
(X[f])Y(-1)0_{P'} &= (X \otimes \frac{1}{w+a})(Y(\frac{1}{w}))0_{P'} \\
&= [(X \otimes \frac{1}{w+a}), Y(\frac{1}{w})]0_{P'} + Y(\frac{1}{w})(X \otimes \frac{1}{w+a})0_{P'} \\
&= [(X \otimes \frac{1}{w+a}), Y(\frac{1}{w})]0_{P'} + Y(\frac{1}{w})(X \otimes \frac{1}{a}(1 - \frac{w}{a} + \frac{w^2}{a^2} - \cdots))0_{P'} \\
&= [(X \otimes \frac{1}{w+a}), Y(\frac{1}{w})]0_{P'} \\
&= \left(\frac{[X,Y]}{a} \otimes w^{-1} - \frac{\ell.(X,Y)}{a^2}\right)0_{P'}.
\end{aligned}
$$

The above computation can be summarized in the form of the following theorem.

Theorem 11. *For a local coordinate z at a nonsingular point P on the curve C we have the following*

$$
\begin{aligned}
&\langle \Psi, X(P)Y(P').X_1(P_1)X_2(P_2)\cdots X_M(P_M).\Phi\rangle \\
= \;& \frac{\ell.(X,Y)}{(z(P)-z(P'))^2})\langle \Psi, X_1(P_1)X_2(P_2)\cdots X_M(P_M).\Phi\rangle \\
&+\frac{1}{z(P)-z(P')}\langle \Psi, [X,Y](P').X_1(P_1)X_2(P_2)\cdots X_M(P_M).\Phi\rangle \\
&+ \textit{regular at } P = P'.
\end{aligned}
$$

Let z be a local coordinate centered at the point Q_i on the curve C and P be a point near Q_i. As before $w = z - z(Q_i)$ be the coordinate centered at the P. For $v \in V_{\vec{\lambda}} = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_N} \subset \mathcal{H}_{\vec{\lambda}}$; where $V_{\lambda_i} = \mathcal{H}_{\lambda_i}(0) = 1 \otimes V(\lambda_i)$. We consider the following:

$$\langle \widehat{\Psi}, X(-1)0_P \otimes \widetilde{v}\rangle dz,$$

where

$$\widetilde{v} = X_1(-1)0_{P_1} \otimes X_2(-1)0_{P_2} \otimes \cdots \otimes X_M(-1)0_{P_M} \otimes v.$$

Consider a function

$$f \in H^0(C, \mathcal{O}_C(*(P + \sum_{i=1}^{N+M} Q_j)))$$

such that

$$f = \frac{1}{w} + \text{regular at P}.$$

We can adjust the coordinate z such that $f = \frac{1}{w}$ at the point P. Also let $a = z(Q_i)$. So the expansion of f around the point Q_i looks like,

$$
\begin{aligned}
f &= \frac{1}{w-a+a} \\
&= \frac{1}{z+a} \\
&= \frac{1}{a(1+\frac{z}{a})} \\
&= \frac{1}{a}(1 - \frac{z}{a} + \frac{z^2}{a^2} - \cdots).
\end{aligned}
$$

We have by gauge symmetry

$$\widehat{\Psi}.X[f] = 0.$$

So we get,

$$\widehat{\Psi}.X[f](0_P \otimes \widetilde{v}) = 0.$$

Thus,

$$\sum_{j=1}^{N+M} \left(\langle \widehat{\Psi}, \rho_j(X[f])\widetilde{v} \otimes 0_P \rangle \right) \quad + \quad \langle \widehat{\Psi}, \widetilde{v} \otimes \rho_P(X[f])0_P \rangle = 0.$$

Hence,

$$\begin{aligned}
&\langle \widehat{\Psi}, \widetilde{v} \otimes \rho_P(X[f])0_P \rangle \\
&\quad = \langle \widehat{\Psi}, \widetilde{v} \otimes X(-1).0_P \rangle \\
&\quad = -\sum_{j=1}^{N+M} \langle \widetilde{\Psi}, \rho_j(X[f])\widetilde{v} \rangle \\
&\quad = -\langle \widetilde{\Psi}, \rho_i(X[f])\widetilde{v} \rangle - \sum_{j \neq i,\ j=1}^{N+M} \langle \widetilde{\Psi}, \rho_j(X[f])\widetilde{v} \rangle.
\end{aligned}$$

The first term on the right hand side of the equation gives $\frac{1}{a}\langle \widetilde{\Psi}, \rho_i(X)\widetilde{v} \rangle$ and the second term is holomorphic in Q_i. Thus the above computation can be summarized in the form of the following theorem.

Theorem 12. *For a local coordinate z at a point Q_i on the curve C we have the following:*

$$\begin{aligned}
&\langle \Psi, X(P).X_1(P_1)X_2(P_2)\cdots X_M(P_M).v \rangle \\
&\quad = \frac{1}{z(P) - z(Q_i)} \langle \Psi, X_1(P_1)X_2(P_2)\cdots X_M(P_M).\rho_i(X).v \rangle \\
&\quad + \textit{regular at } P = Q_i.
\end{aligned}$$

The function F is called the *correlation function of currents* and plays an important role in the decomposition of conformal blocks. We started with a simple looking meromorphic form and it turns from our above discussion that the poles of that form is rather well behaved. In the next section we compute the correlation function for the projective line $\mathbb{P}^1$ and a one pointed elliptic curve. We end this section by stating a theorem without proof which tell us about the connection of this form with the Energy Momentum Tensor.

Define

$$\begin{aligned}
\langle \Psi, T(z)\Phi \rangle dz^2 = \frac{1}{2(g^* + \ell)} \lim_{z \to w} \Bigg(&\sum_{a=1}^{\dim \mathfrak{g}} \langle \Psi, J^a(z)J^a(w).\Phi \rangle dzdw \\
&- \frac{\ell . \dim \mathfrak{g}}{(z-w)^2} \langle \Psi, \Phi \rangle dzdw \Bigg).
\end{aligned}$$

where $J^1, J^2, \cdots, J^a, \cdots, J^{\dim \mathfrak{g}}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the normalized Cartan Killing form $\langle \ , \ \rangle$. Then the following holds and can be proved by direct computation:

Proposition 6.

$$\operatorname*{Res}_{\xi_k=0}\left({\xi_k}^{n+1}\langle\Psi, T(\xi_k)\Phi\rangle\right)d\xi_k = \langle\Psi, \rho_k(L_n)\Phi\rangle.$$

So we have an expansion,

$$\langle\Psi, \Phi\rangle d{\xi_k}^2 = \sum_{n\in\mathbb{Z}}\langle\Psi, \rho_k(L_n).\Phi\rangle{\xi_k}^{-n-2}d{\xi_k}^2.$$

4.2 Correlation functions for N-pointed projective line $\mathbb{P}^1$

Let z be a global coordinate of $\mathbb{C}$ and $z_1, z_2, z_3, \cdots, z_N$ be N points on the projective space $\mathbb{P}^1$ and ξ_i be the local-coordinate at the point z_i. In this case we can explicitly write ξ_i as

$$\xi_i = \begin{cases} z - z_i & \text{if } z_i \neq \infty, \\ \frac{1}{z} & \text{if } z_i = \infty. \end{cases}$$

Let

$$\mathfrak{X} = (\mathbb{P}^1; z_1, \cdots, z_N; \xi_1, \xi_2, \cdots, \xi_N)$$

be the data associated to the N-pointed projective line $\mathbb{P}^1$.

Proposition 7. *Let* $V_{\vec{\lambda}} = (V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_N})$, *where*

$$\vec{\lambda} = (\lambda_1, \cdots, \lambda_N) \in P_\ell^N.$$

The restriction map

$$Hom_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C}) \to Hom_{\mathbb{C}}(V_{\vec{\lambda}}, \mathbb{C})$$

induces an injective homomorphism

$$\iota : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \hookrightarrow Hom_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C}).$$

Proof. First we show that we land up in the right target space. For $\Psi \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ and $X \in \mathfrak{g}$ we have

$$\sum_{j=1}^{N}\Psi.\rho_j(X \otimes 1) = 0.$$

Since the action of $\mathfrak{g}$ on a 1 dimensional $\mathfrak{g}$-module is trivial, we get the image ι lies in $\mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C})$.

If $\iota(\Psi) = 0$, by induction on $F_p\mathcal{H}_{\vec{\lambda}}$, we show that Ψ restricted to $F_p\mathcal{H}_{\vec{\lambda}}$ is 0.

For $p = 0$, this is true by the assumption $\iota(\Psi) = 0$. Now by induction, assume Ψ restricted to $F_p\mathcal{H}_{\vec{\lambda}} = 0$. Now any element Φ of $F_{p+1}\mathcal{H}_{\vec{\lambda}}$ can be written as

$$\rho_j(X(-n)).v, \quad v \in F_p(\mathcal{H}_{\vec{\lambda}}),$$

for some j and a positive integer n. Consider the function

$$f = \frac{1}{(z - z_j)^n}.$$

Observe that these function f does not have any other poles except at the point $z = z_j$. Then,

$$\begin{aligned}
\langle \Psi, \Phi \rangle &= \langle \Psi, \rho_j(X(-n)).v \rangle \\
&= \langle \Psi, \rho_j(X \otimes f).v \rangle \\
&= - \sum_{i=1, i \neq j}^{N} \langle \Psi, \rho_i(X \otimes f).v \rangle \\
&= 0 \quad (\text{ By induction }).
\end{aligned}$$

□

We list down some computation on the dimension of conformal blocks on $\mathbb{P}^1$.

(1) For $N = 1$ and $\lambda \neq 0$, we have

$$\mathcal{V}^{\dagger}_{\lambda}(\mathfrak{X}) = 0.$$

(2) For $N = 2$, we have

$$\mathcal{V}^{\dagger}_{\lambda,\mu}(\mathfrak{X}) = \mathbb{C} \quad if \ \mu = \lambda^{\dagger},$$

other wise 0.

(3) For $N = 3$ and $\vec{\lambda} = (\mu, \nu, \lambda)$, we have

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}) \simeq W_{\vec{\lambda}}.$$

where

$$W_{\vec{\lambda}} = \{\psi \in \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}.\mathbb{C}) | \ \text{ condition } (*) \ \},$$

The condition $(*)$ is defined below. Let

$$\mathfrak{t} = \mathbb{C}X_{\theta} \oplus \mathbb{C}X_{-\theta} \oplus \mathbb{C}[X_{\theta}, X_{-\theta}]$$

be the principal three-dimensional subalgebra of $\mathfrak{g}$.

$$V_{\lambda} = \bigoplus_{i=0}^{\ell/2} W_{\lambda,i}$$

be the irreducible decomposition as a $\mathfrak{t}$-module. Then the condition $(*)$ is given by,

$$\Psi|_{W_{\mu,k}\otimes W_{\nu,i}\otimes W_{\lambda,j}} = 0, \quad if\ k+i+j > \ell.$$

For a proof of above we refer to [Beau].

There is no holomorphic one form on $\mathbb{P}^1$. For $X \in \mathfrak{g}$ and $v \in \mathcal{H}_{\vec{\lambda}}(0)$, if none of the points are ∞, we get from theorem 12

$$\langle \Psi, X(z).v\rangle dz = \sum_{j=1}^{N} \frac{1}{z-z_j}\langle \Psi, \rho_j(X).v\rangle dz.$$

We compute the residue at the points z_i (z_i could be ∞) of $\langle \Psi, X(z).v\rangle dz$. We know that the form can be expanded in local coordinates around z_i as

$$\sum_{n\in\mathbb{Z}} \langle \Psi, \rho_i(X(n)).v\rangle \xi_i^{-n-1} d\xi_i.$$

The residue at z_i is

$$\operatorname*{Res}_{\xi_i=o} \langle \Psi, X(\xi_i).v\rangle d\xi_i = \langle \Psi, \rho_i(X).v\rangle.$$

The sum of the residues is

$$\sum_{i=1}^{N} \langle \Psi, \rho_i(X).v\rangle = \Psi.X(v).$$

By gauge symmetry, we know that $\Psi.X = 0$. The expression

$$\frac{1}{z-z_j}\langle \Psi, \rho_j(X).v\rangle dz$$

does not have poles at any other point z_k, where $k \neq j$. This is true since

$$\begin{aligned} \frac{1}{z-z_j} &= \frac{1}{z-z_k+z_k-z_j} \\ &= \frac{1}{z_k-z_j}\left(1 - \frac{z-z_k}{z_k-z_j} + (\frac{z-z_k}{z_k-z_j})^2 + \cdots\right) \end{aligned}$$

is holomorphic. Now if one of the points say z_1 is infinity, we consider the form

$$\omega = \langle \Psi, X(z).v\rangle dz - \sum_{j=2}^{N} \frac{1}{z-z_j}\langle \Psi, \rho_j(X).v\rangle dz.$$

The residue of the form $\sum_{j=2}^{N} \frac{1}{z-z_j}\langle \Psi, \rho_j(X).v\rangle dz$ at infinity is $-\sum_{j=2}^{N}\langle \Psi, \rho_j(X).v\rangle$. But by gauge symmetry, we know it is same as $\langle \Psi, \rho_1(X).v\rangle$. We have already shown that the residue of the form $\langle \Psi, X(z).v\rangle dz$ at the point $z_1 = \infty$ is also $\langle \Psi, \rho_1(X).v\rangle$. Thus the residues of the form ω at all the z_i's is zero. Now since there are no holomorphic 1-forms on $\mathbb{P}^1$, we get

$$\langle \Psi, X(z).v\rangle dz - \sum_{j=2}^{N} \frac{1}{z-z_j}\langle \Psi, \rho_j(X).v\rangle dz = 0.$$

Let $X, Y \in \mathfrak{g}$ and $\Psi \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$ and $v \in \mathcal{H}_{\vec{\lambda}}(0)$. Now let $a = z(P) - z(w)$. Modify it a little bit to get $a = z(P) - z + z - z(P')$, where z is centered at the point P. So $z(P) = 0$ and $w = z - z(P')$ is a coordinate centered around P'. Thus, we get $a = z - w$. From the theorem 11 on correlation functions we get,

$$\begin{aligned}\langle \Psi, X(z)Y(w).v\rangle dzdw &= \frac{\ell\langle X, Y\rangle}{(z-w)^2}\langle \Psi, v\rangle dzdw + \frac{1}{z-w}\langle \Psi, [X,Y](w).v\rangle dzdw \\ &+ \text{ poles at } z = z_j.\end{aligned}$$

$\langle \Psi, X(z)Y(w).v\rangle dzdw$ has simple poles at the points $z = z_j$. Assume that none of the z_j's are infinity. The residue of the form at z_j is $\langle \Psi, Y(-1)0_{P'} \otimes \rho_j(X)v\rangle$. Since there are no holomorphic one forms on $\mathbb{P}^1 \times \mathbb{P}^1$, we get the form

$$\langle \Psi, X(z)Y(w).v\rangle dzdw$$

looks like,

$$\begin{aligned}\langle \Psi, X(z)Y(w).v\rangle dzdw &= \frac{\ell\langle X, Y\rangle}{(z-w)^2}\langle \Psi, v\rangle dzdw + \frac{1}{z-w}\langle \Psi, [X,Y](w).v\rangle dzdw \\ &+ \sum_{j=1}^{N} \frac{1}{z-z_j}\langle \Psi, Y(w).\rho_j(X).v\rangle dzdw.\end{aligned}$$

If one of the z_i say $z_1 = \infty$, we get by gauge symmetry and computing residues.

$$\begin{aligned}\langle \Psi, X(z)Y(w).v\rangle dzdw &= \frac{\ell\langle X, Y\rangle}{(z-w)^2}\langle \Psi, v\rangle dzdw + \frac{1}{z-w}\langle \Psi, [X,Y](w).v\rangle dzdw \\ &+ \sum_{j=2}^{N} \frac{1}{z-z_j}\langle \Psi, Y(w).\rho_j(X).v\rangle dzdw.\end{aligned}$$

Now we are interested in the special case when $X = Y$. Using the above computations we get,

$$\begin{aligned}\langle \Psi, X(z)X(w).v\rangle dzdw &= \frac{\ell\langle X, X\rangle}{(z-w)^2}\langle \Psi, v\rangle dzdw + \frac{1}{z-w}\langle \Psi, [X,X](w).v\rangle dzdw \\ &\quad + \sum_{j=1}^{N} \frac{1}{z-z_j}\langle \Psi, X(w).\rho_j(X).v\rangle dzdw \\ &= \frac{\ell\langle X, X\rangle}{(z-w)^2}\langle \Psi, v\rangle dzdw \\ &\quad + \sum_{k=1}^{N}\sum_{j=1}^{N} \frac{1}{(z-z_j)(z-z_k)}\langle \Psi, \rho_k(X)\rho_j(X)v\rangle dzdw.\end{aligned}$$

By the definition of the correlation function for the energy momentum tensor we conclude,

$$\langle \Psi, T(z).v\rangle dz^2 = \frac{1}{2(g^*+\ell)} \sum_{k=1}^{N}\sum_{j=1}^{N} \frac{1}{(z-z_j)(z-z_k)} \sum_{a=1}^{\dim \mathfrak{g}} \langle \Psi, \rho_k(J^a)\rho_j(J^a)v\rangle dz^2.$$

4.3 Correlation function for 1-pointed elliptic curve

Let z be a global coordinate of the complex plane $\mathbb{C}$ and 0 be the origin. For simplicity let 0 be the marked point of the elliptic curve. Since the genus of elliptic curve is 1, we know that there is only one global holomorphic one form on the elliptic curve generated by $\mathbb{C}dz$.

Also we know from complex analysis that there is no meromorphic function on the elliptic curves with a single simple pole. The Weierstrass $\wp$ function is one such function which has a pole of order two at the origin and is given by the following expression:

$$\wp_\Gamma(z) = \frac{1}{z^2} + \sum_{\omega\in\Gamma-0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right);$$

where Γ is the lattice such that $E = \mathbb{C}/\Gamma$. Let $\mathfrak{X}$ is the data associated to 1-pointed Elliptic curve. For $X \in \mathfrak{g}$, $v \in \mathcal{H}_\lambda(0)$ and $\Psi \in \mathcal{V}_\lambda^\dagger(\mathfrak{X})$, we consider the meromorphic one form $\langle \Psi, X(z).v\rangle dz$. By the theorem 9 we know that this has poles only at the origin. Taking the Laurent series expansion around the origin, we can write it as

$$\begin{aligned}\langle \Psi, X(z)v\rangle dz &= \sum_{n\in\mathbb{Z}} \langle \Psi, X(n)v\rangle z^{-n-1}dz \\ &= \sum_{n\geq 0} \langle \Psi, X(-n)v\rangle z^{n-1}dz\end{aligned}$$

Since the meromorphic form cannot have a single simple pole at the origin, we have $\langle \Psi, X.v\rangle = 0$. This implies that the form is holomorphic. By our remark, we know that the form should look like kdz; where $k \in \mathbb{C}$. So this implies $\langle \Psi, X(-n).v\rangle = 0$ if $n \neq 1$. Thus, we get

$$\langle \Psi, X(z)v\rangle dz = \langle \Psi, X(-1)v\rangle dz.$$

Let $X \in \mathfrak{g}$ and w be a global coordinate centered at the point $P \neq 0$. We want to compute the form $\langle \Psi, X(z)X(w).v\rangle dzdw$. From the proof of theorem 11, we get

$$\langle \Psi, X(-1)0_P \otimes X(-1)0_{P'} \otimes v\rangle = \frac{\ell.\langle X, X\rangle}{a^2}\langle \Psi, v\rangle - \langle \Psi, X(-1)0_{P'} \otimes X(-1)v\rangle.$$

Now let us consider the term $\langle \Psi, X(-1)0_{P'} \otimes X(-1)v\rangle$. By gauge symmetry, this is same as $-\langle \Psi, X(-1).X(-1)v\rangle$. Thus we get

$$\langle \Psi, X(z)X(w).v\rangle dzdw = \frac{\ell.\langle X, X\rangle}{(z-w)^2}\langle \Psi, v\rangle dzdw + \langle \Psi, X(-1).X(-1).v\rangle dzdw.$$

From the definition of $\langle \Psi, T(z).v\rangle dz^2$ and using the above computation

$$\langle \Psi, T(z).v\rangle dz^2 = \frac{1}{2(g^* + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} \langle \Psi, J^a(-1).J^a(-1)v\rangle dz^2.$$

Thus we see that the form is actually homomorphic and has no poles at 0.

Again by proposition-6

$$\langle \Psi, T(z).v\rangle dz^2 \;\; = \;\; \sum_{n\in\mathbb{Z}} \langle \Psi, L_n v\rangle z^{-n-2} dz^2.$$

Now for $n > 0$, we know that $L_n.v = 0$. So the above expression can be rewritten as,

$$\begin{aligned} \langle \Psi, T(z).v\rangle dz^2 &= \sum_{n\leq 0} \langle \Psi, L_n v\rangle z^{-n-2} dz^2 \\ &= \langle \Psi, L_0.v\rangle z^{-2} dz^2 + \langle \Psi, L_{-1}.v\rangle z^{-1} dz^2 + \langle \Psi, L_{-2}.v\rangle dz^2 \cdots \end{aligned}$$

Since the form is holomorphic, we get

$$\langle \Psi, T(z).v\rangle dz^2 = \langle \Psi, L_{-2}.v\rangle dz^2.$$

Chapter 5

Decomposition of Conformal Blocks

5.1 Factorization

In this section we describe a decomposition of conformal blocks. We state and give a proof of the *Factorization theorem* of conformal blocks. The conformal block of a pointed stable curve is intimately connected with the normalization of the curve. We start with a N-pointed stable curve C and we have the following data.

$$\mathfrak{X} = (C; Q_1, Q_2, \cdots, Q_N; \eta_1, \eta_2 \cdots, \eta_N).$$

For simplicity assume that the curve C has a single node P. Let $\widetilde{C}$ be its normalization and

$$\nu : \widetilde{C} \longrightarrow C$$

such that $\nu^{-1}(P) = \{P', P''\}$.

Associated to a curve $\widetilde{C}$, we have another data

$$\widetilde{\mathfrak{X}} = (\widetilde{C}; P', P'', Q_1, Q_2, \cdots, Q_N; \eta', \eta'', \eta_1, \eta_2 \cdots, \eta_N);$$

where η' and η'' are formal parameter at the new added points. For $\mu \in P_\ell$ and w, the longest element of the Weyl Group, we set

$$\mu^\dagger = -w(\mu).$$

We also have

$$w(\Delta^+) = \Delta^-.$$

For the new added points P' and P'' to the normalization $\widetilde{C}$ of the N-pointed nodal curve C, consider the representation μ and $\mu^\dagger$; where $\mu \in P_\ell$. The conformal block for the above data is denoted by

$$\mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}).$$

We would like to connect it with the original conformal block $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ associated to the data $\mathfrak{X}$. We have the following theorem which gives a decomposition of conformal blocks over a curve into conformal blocks over a curve with lower genus.

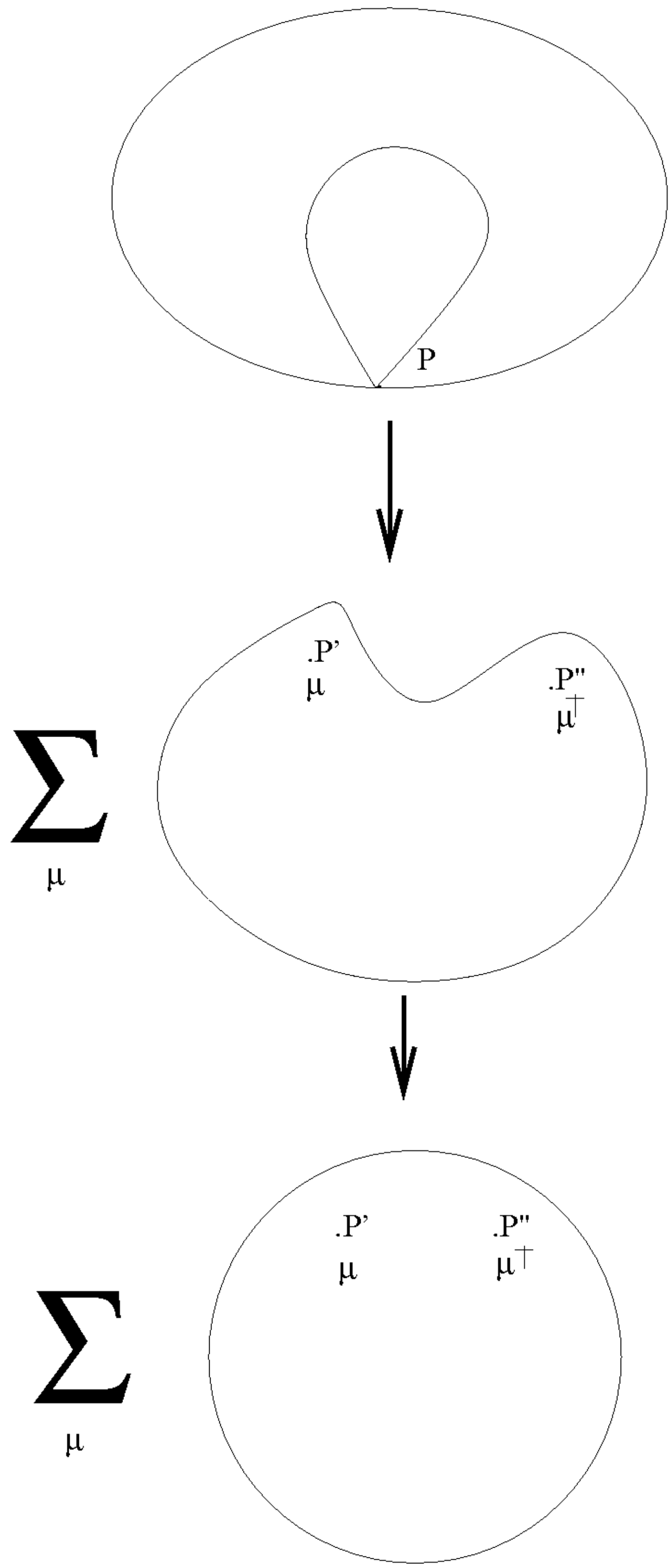

Figure 5.1: Normalization of a curve at P

Theorem 13. *(Factorization)*

For a N-pointed stable curve C and the following data

$$\mathfrak{X} = (C; Q_1, Q_2, \cdots, Q_N; \eta_1, \eta_2 \cdots, \eta_N).$$

Let $\widetilde{C}$ be its normalization and we the following data

$$\widetilde{\mathfrak{X}} = (\widetilde{C}; P', P'', Q_1, Q_2, \cdots, Q_N; \eta', \eta'', \eta_1, \eta_2, \cdots, \eta_N).$$

We have a canonical isomorphism

$$\bigoplus_{\mu \in P_\ell} \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}).$$

We prove the theorem in various steps. First we construct injective maps

$$\widehat{\iota}_\mu : \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}) \to \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}).$$

Then we paste these maps together to get a map

$$\widehat{\iota} = \oplus \widehat{\iota}_\mu : \bigoplus_{\mu \in P_\ell} \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}).$$

Then we show the map $\widehat{\iota}$ is injective and surjective respectively. Let us start by defining the map $\widehat{\iota}_\mu$. Let $0_{\mu,\mu^\dagger}$ be the generator of the $\mathfrak{g}$-module $V_\mu \otimes V_{\mu^\dagger}$. The action of $\mathfrak{g}$ on $0_{\mu,\mu^\dagger}$ is given by

$$\rho_1(X)0_{\mu,\mu^\dagger} + \rho_2(X)0_{\mu,\mu^\dagger} = 0;$$

where X is an arbitrary element of the Lie algebra $\mathfrak{g}$. The $\widehat{\mathfrak{g}}$-module $\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}}$ contains $0_{\mu,\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}$, i.e.

$$\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}} \supset 0_{\mu,\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}} \simeq \mathcal{H}_{\vec{\lambda}}.$$

Given an element of $\widetilde{\Psi} \in \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}})$, we define $\Psi \in \mathcal{H}_{\vec{\lambda}}{}^\dagger$ by

$$\langle \Psi, \Phi \rangle = \langle \widetilde{\Psi}, 0_{\mu,\mu^\dagger} \otimes \Phi \rangle.$$

If we can show that this element Ψ respects the gauge condition, then we can define $\widehat{\iota}_\mu(\widetilde{\Psi}) = \Psi$. For all $f \in H^0(C, \mathcal{O}_C(* \sum_{j=1}^N Q_j))$, we need to show that

$$\Psi.X[f] = \sum_{j=1}^N \Psi.\rho_j(X[f]) = 0$$

on $\mathcal{H}_{\vec{\lambda}}$. The function f can be considered as an element of $H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(* \sum_{j=1}^N Q_j))$ such that $f(P') = f(P'')$. The diagonal action of $\mathfrak{g}$ gives,

$$\rho_{P'}(X[f])0_{\mu,\mu^\dagger} + \rho_{P''}(X[f])0_{\mu,\mu^\dagger} = 0.$$

Thus

$$\begin{aligned}\sum_{j=1}^{N}\langle\Psi,\rho_j(X[f])\Phi\rangle &= \sum_{j=1}^{N}\langle\widetilde{\Psi},\rho_j(X[f]).0_{\mu,\mu^\dagger}\otimes\Phi\rangle\\ &= \sum_{j=1}^{N+2}\langle\widetilde{\Psi},\rho_j(X[f]).0_{\mu,\mu^\dagger}\otimes\Phi\rangle\\ &= \widetilde{\Psi}.X[f](0_{\mu.\mu^\dagger}\otimes\Phi)\\ &= 0 \quad (\text{ By gauge symmetry }).\end{aligned}$$

We have proved that following,

Lemma 18. *Ψ is an element of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ and we can define $\widehat{\iota}_\mu(\widetilde{\Psi})=\Psi$.*

Lemma 19. *The map*

$$\widehat{\iota}_\mu:\mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}})\longrightarrow\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$$

is injective.

Proof. We would like to consider the form $\langle\Psi,X(P_1).\Phi\rangle dP_1$ as a form on $\widetilde{C}$. We claim that it is the same as the form $\langle\widetilde{\Psi},X(P_1).0_{\mu,\mu^\dagger}\otimes\Phi\rangle dP_1$. For that we need to check that at each Q_j, the local expansion of the above two forms are the same.

Using theorem 9 on correlation functions, we see that $\langle\Psi,X(P_1).\Phi\rangle dP_1$ has a local expansion at Q_j with respect to the formal parameter ξ_j

$$\sum_{n\in\mathbb{Z}}\langle\Psi,\rho_j(X(n))\Phi\rangle\xi_j{}^{-n-1}d\xi_j.$$

Using theorem 9 again, we see that around Q_j the form $\langle\widetilde{\Psi},X(P_1).0_{\mu,\mu^\dagger}\otimes\Phi\rangle dP_1$ has an expansion

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\langle\widetilde{\Psi},\rho_j(X(n)).0_{\mu,\mu^\dagger}\otimes\Phi\rangle\xi_j{}^{-n-1}d\xi_j &= \sum_{n\in\mathbb{Z}}\langle\widetilde{\Psi},0_{\mu,\mu^\dagger}\otimes\rho_j(X(n))\Phi\rangle\xi_j{}^{-n-1}d\xi_j\\ &= \sum_{n\in\mathbb{Z}}\langle\Psi,\rho_j(X(n)).\Phi\rangle\xi_j{}^{-n-1}d\xi_j.\end{aligned}$$

Repeating the same argument, one can show that the form

$$\langle\Psi,X_1(P_1)X_2(P_2)\cdots X_N(P_N).\Phi\rangle dP_1dP_2\cdots dP_N$$

considered as a form on the curve $\widetilde{C}$ is same as the form

$$\langle\Psi,X_1(P_1)X_2(P_2)\cdots X_N(P_N).0_{\mu,\mu^\dagger}\otimes\Phi\rangle dP_1dP_2\cdots dP_N.$$

Now if we assume that $\Psi=0$, we see the form

$$\langle\Psi,X_1(P_1)X_2(P_2)\cdots X_N(P_N).0_{\mu.\mu^\dagger}\otimes\Phi\rangle dP_1dP_2\cdots dP_N$$

is also zero.

So the local expansion around the point P' and P'' is also zero. Using the expansion of correlation functions we see,

$$\langle \widetilde{\Psi}, X_2(P_2) \cdots .X_N(P_N).\rho_{P'}(X_1(n))0_{\mu,\mu^\dagger} \otimes \Phi \rangle = 0.$$

Repeating the above process, we have for integers m and n and $X_i \in \mathfrak{g}$, we have the following:

$$\begin{aligned}
\langle \widetilde{\Psi}, \rho_{P'}(X_1(n))\rho_{P'}(X_2(m))0_{\mu,\mu^\dagger} \otimes \Phi \rangle &= 0. \\
\langle \widetilde{\Psi}, \rho_{P'}(X_1(n))\rho_{P''}(X_2(m))0_{\mu,\mu^\dagger} \otimes \Phi \rangle &= 0. \\
\langle \widetilde{\Psi}, \rho_{P''}(X_1(n))\rho_{P''}(X_2(m))0_{\mu,\mu^\dagger} \otimes \Phi \rangle &= 0.
\end{aligned}$$

Repeating the above process, we get $\langle \widetilde{\Psi}, \widetilde{\Phi} \rangle = 0$ where $\widetilde{\Phi} \in \mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}}$. This is possible since $\mathcal{H}_\mu \otimes \mathcal{H}_{\mu^\dagger}$ is an irreducible $\widehat{\mathfrak{g}} \times \widehat{\mathfrak{g}}$-module. So any element of $\mathcal{H}_\mu \otimes \mathcal{H}_{\mu^\dagger}$ can be generated by multiplying on the left of $0_{\mu,\mu^\dagger}$ by elements $\rho_1(X(m))$, $\rho_2(Y(n))$; where X, Y are arbitrary elements in $\mathfrak{g}$ and m, n are integers. Repeating the same process, we get $\langle \widetilde{\Psi}, \widetilde{\Phi} \rangle = 0$, where $\widetilde{\Phi} \in \mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}}$. So we have a $\mathbb{C}$-linear injection $\mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}})$ to $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ via the map $\widehat{\iota}_\mu$.

□

For every $\mu \in P_\ell$ we have a map of the form $\widehat{\iota}_\mu$. We paste all these maps together to we get a $\mathbb{C}$-linear map

$$\widehat{\iota} = \oplus \widehat{\iota}_\mu : \bigoplus_{\mu \in P_\ell} \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}) \longrightarrow \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}).$$

We will show in steps that the map $\widehat{\iota}$ is an isomorphism. Before we proceed, we associate certain right $\mathfrak{g}$-module and highest weight integrable right $\widehat{\mathfrak{g}}$-module to the points P' and P''.

We fix an element $\Psi \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$. Let $h \in H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(* \sum_{j=1}^N Q_j))$ such that

$$h(P') = 1 \quad h(P'') = 0.$$

For $u \in \mathcal{H}_{\vec{\lambda}}$, consider the following expression

$$\sum_{j=1}^N \langle \Psi, \rho_j(X[h]).u \rangle.$$

Let $\widetilde{h}$ be another function satisfying the same conditions as that of h. Then $h - h'$ can be considered as an element of $H^0(C, \mathcal{O}_C(* \sum_{j=1}^N Q_j))$. By the gauge condition, we get

$$\sum_{j=1}^N \langle \Psi, \rho_j(X[h]).u \rangle = \sum_{j=1}^N \langle \Psi, \rho_j(X[\widetilde{h}]).u \rangle.$$

Hence we have shown that the expression $\sum_{j=1}^N \langle \Psi, \rho_j(X[h]).u \rangle$ is independent of the choice of the function h satisfying the specified conditions.

Now, for each $X \in \mathfrak{g}$, we define an element $\Psi.\rho_{P'}(X) \in \mathrm{Hom}_{\mathbb{C}}(V_{\vec{\lambda}}, \mathbb{C})$,

$$(\Psi.\rho_{P'}(X))(u) = -\sum_{j=1}^{N} \langle \Psi, \rho_j(X[h]).u \rangle;$$

where $V_{\vec{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$ and let $u \in V_{\vec{\lambda}}$. The above is well defined since we have proved that the right hand side of the expression is well defined. For $X, Y \in \mathfrak{g}$ we define the element

$$(\Psi.\rho_{P'}(X)\rho_{P'}(Y))(u) = \sum_{j=1}^{N}\sum_{k=1}^{N} \langle \Psi, \rho_k(Y[h_2])\rho_j(X[h_1]).u \rangle;$$

where h_1 and h_2 satisfy similar conditions as that of the function h. From the previous argument it is clear that the above expression is independent of the choice of the function h_2. We need to show that it is independent of the choice of the function h_1. For that consider the form $h_2 dh_1$. This is a meromorphic form on the curve $\widetilde{C}$ and has poles only at the point Q_j. Thus, we get

$$\sum_{j=1}^{N} \mathrm{Res}_{Q_j}(h_2 dh_1) = 0.$$

$$\begin{aligned}\sum_{j=1}^{N}\sum_{k=1}^{N} \langle \Psi, \rho_k(Y[h_2])\rho_j(X[h_1]).u \rangle &= \sum_{j=1}^{N}\sum_{k=1}^{N} \langle \Psi, \rho_j(X[h_1])\rho_k(Y[h_2]).u \rangle \\ &\quad - \sum_{j=1}^{N} \langle \Psi, \rho_j([X,Y](h_1h_2)).u \rangle.\end{aligned}$$

The right hand side is well defined hence the left hand side is independent of the choice of the functions h_1 and h_2. Also we observe from the equality above that,

$$(\Psi.(\rho_{P'}(X)\rho_{P'}(Y) - \rho_{P'}(Y)\rho_{P'}(X))) = (\Psi.\rho_{P'}([X,Y])).$$

So we have defined a right $\mathfrak{g}$-module $U(\Psi)$ at the point P'. Similarly we can define a right $\mathfrak{g}$-module at the point P''.

Using similar techniques we will define an integrable right $\widehat{\mathfrak{g}}$-module at the point P'. For any integer n, we define an element $\Psi.\rho_{P'}(X(n)) \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C})$ as follows:

Let g be a meromorphic function on the curve $\widetilde{C}$ such that for a large integer M we have

$$\begin{aligned} g &\in H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(*\sum_{j=1}^{N} Q_j))' \\ g &= \xi'^n \mod (\xi'^M) \ at\ P', \\ g(P'') &= 0; \end{aligned}$$

where $\xi' = \eta^{-1}(\xi)$ is a formal parameter at the point P'. For any $u \in \mathcal{H}_{\vec{\lambda}}$, we define

$$(\Psi\rho_{P'}(X(n))).u = -\sum_{j=1}^{N}\langle\Psi, \rho_j(X[g]).u\rangle.$$

By previous type of arguments, it is clear that the expression is independent of the choice of the function g satisfying the above mentioned properties.

We define

$$(\Psi.\rho_{P'}(X(n))\rho_{P'}(Y(m)))(u) = \sum_{j=1}^{N}\sum_{k=1}^{N}\langle\Psi, \rho_k(Y[g_2])\rho_j(X[g_1]).u\rangle,$$

where g_1 and g_2 satisfy "similar" conditions as that of the function g. This is independent of the function g_1 and g_2 satisfying the above properties. We also have the following equality:

$$\begin{aligned}\sum_{j=1}^{N}\sum_{k=1}^{N}\langle\Psi, \rho_k(Y[g_2])\rho_j(X[g_1]).u\rangle \quad &= \quad \sum_{j=1}^{N}\sum_{k=1}^{N}\langle\Psi, \rho_j(X[g_1])\rho_k(Y[g_2]).u\rangle \\ -\sum_{j=1}^{N}\langle\Psi, \rho_j([X,Y](h_1h_2)).u\rangle \quad &- \quad \ell.\langle X, Y\rangle n\delta_{n+m,0}\langle\Psi, u\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}\Big(\Psi.\big(\rho_{P'}(X(n))\rho_{P'}(Y(m)) \quad &- \quad \rho_{P'}(Y(m))\rho_{P'}(X(n))\big)\Big) \\ &= \quad (\Psi.\rho_{P'}([X,Y](m+n))) + \ell.\langle X, Y\rangle n\delta_{n+m,0}(\Psi).\end{aligned}$$

So we have constructed a right $\widehat{\mathfrak{g}}$-module $\widehat{\mathcal{U}}(\Psi)$. Since the action of $\widehat{\mathfrak{g}}$ on $\mathcal{H}_{\vec{\lambda}}$ is locally nilpotent, the action of $\rho_{P'}(X(m))$ is also locally nilpotent. Hence $\widehat{\mathcal{U}}(\Psi)$ is an integrable right $\widehat{\mathfrak{g}}$-module of level ℓ.

We define a finite dimensional right $\mathfrak{g}$-module at the point P'

$$\mathcal{U}(\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})) = \bigcup_{\Psi \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})} \mathcal{U}(\Psi) \subset \mathrm{Hom}_{\mathbb{C}}(V_{\vec{\lambda}}, \mathbb{C})$$

and an integrable right $\widehat{\mathfrak{g}}$-module at the point P' as

$$\widehat{\mathcal{U}}(\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})) = \bigcup_{\Psi \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})} \widehat{\mathcal{U}}(\Psi) \subset \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C}).$$

Using complete reducibility, we get

$$\mathcal{U}(\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})) = \bigoplus_{\mu \in P_\ell} V_{\mu}^{\dagger n_\mu},$$

where n_μ is the multiplicity of the representation V_μ. We also have

$$\widehat{\mathcal{U}}(\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})) = \bigoplus_{\mu \in P_\ell} {\mathcal{H}^\dagger_\mu}^{n_\mu}.$$

We are ready to prove the injectivity of the map $\widehat{\iota}$.

Lemma 20. *The $\mathbb{C}$-linear map*

$$\widehat{\iota}: \bigoplus_{\mu \in P_\ell} \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}}) \longrightarrow \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$$

is injective.

Proof. Using the fact that the maps $\widehat{\iota}_\mu$ are injective, we only need to show that the images of $\widehat{\iota}_\mu$ and $\widehat{\iota}_\nu$ do not meet. Let $\Psi = \widehat{\iota}_\mu(\widetilde{\Psi})$. We choose a meromorphic function $h \in H^0(\widetilde{C}.\mathcal{O}_{\widetilde{C}}(* \sum_{j=1}^N *Q_j))$ on $\widetilde{C}$, such that $h(P') = 1$ and $h(P'') = 0$.

By gauge symmetry, we have

$$\begin{aligned}
&\langle \widetilde{\Psi}, \rho_{P'}(X_1(0)) \cdots \rho_{P'}(X_k(0)).0_{\mu,\mu^\dagger} \otimes u\rangle \\
&= (-1) \sum_{j_1=1}^N \langle \widetilde{\Psi}, \rho_{P'}(X_2(0)) \cdots \rho_{P'}(X_k(0)).0_{\mu,\mu^\dagger} \otimes \rho_{j_1}(X[h]).u\rangle \\
&= (-1)^2 \sum_{j_1=1, j_2=1}^N \langle \widetilde{\Psi}, \rho_{P'}(X_3(0)) \cdots \rho_{P'}(X_k(0)).0_{\mu,\mu^\dagger} \otimes \rho_{j_2}(X[h])\rho_{j_1}(X[h]).u\rangle \\
&= (-1)^{k+1} \sum_{j_1=1, j_2=1, \cdots, j_k=1}^N \langle \widetilde{\Psi}, 0_{\mu,\mu^\dagger} \otimes \rho_{j_k}(X_1[h]) \cdots \rho_{j_1}(X_k[h]).u\rangle \\
&= (-1)^{k+1} \sum_{j_1=1, j_2=1, \cdots, j_k=1}^N \langle \Psi, \rho_{j_k}(X_1[h]) \cdots \rho_{j_1}(X_k[h]).u\rangle \\
&= (-1)^{k+1} \sum_{j_1=1, j_2=1, \cdots, j_k=1}^N \langle \Psi, \rho_{j_1}(X_1[h]) \cdots \rho_{j_k}(X_k[h]).u\rangle.
\end{aligned}$$

The last equality follows from the fact that the $\rho_{j_i}(X[h])$ commutes with $\rho_{j_k}(X[h])$.

Then by definition for a fixed $u \in \mathcal{H}_{\vec{\lambda}}$,

$$\begin{aligned}
&(\Psi.\rho_{P'}(X_1).\rho_{P'}(X_2).\cdots \rho_{P'}(X_k))(u) \\
&\quad = (-1)^k \sum_{j_1=1, j_2=1, \cdots, j_k=1}^N \langle \Psi, \rho_{j_1}(X_1[h]) \cdots \rho_{j_k}(X_k[h]).u\rangle.
\end{aligned}$$

Hence,

$$(\Psi.\rho_{P'}(X_1).\rho_{P'}(X_2).\cdots \rho_{P'}(X_k))(u) = (-1)^{k+1}\langle \widetilde{\Psi}, \rho_{P'}(X_1(0)) \cdots \rho_{P'}(X_k(0)).0_{\mu,\mu^\dagger} \otimes u\rangle.$$

Now the elements $\rho_{P'}(X_1(0))\cdots\rho_{P'}(X_k(0)).0_{\mu,\mu^\dagger}$ generate an irreducible left $\mathfrak{g}$-module isomorphic to V_μ. Thus we conclude

$$\mathcal{U}(\Psi)\subset V_\mu^{\dagger n_\mu}.$$

Now for $\widetilde{\Psi}_\mu \in \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}})$, $\widetilde{\Psi}_\nu \in \mathcal{V}^\dagger_{\nu,\nu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}})$ and $\mu\neq\nu$, we have

$$\mathcal{U}(\widehat{\iota}_\mu\widetilde{\Psi}_\mu)\cap\mathcal{U}(\widehat{\iota}_\nu\widetilde{\Psi}_\nu)=0.$$

This means that there images do not intersect. Since $\widehat{\iota}_\mu$'s are injective, $\widehat{\iota}$ is also injective. □

Lemma 21. *The map*

$$\widehat{\iota}\colon\bigoplus_{\mu\in P_\ell}\mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}})\longrightarrow\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$$

is surjective.

Proof. Given an element $\Psi\in\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ we consider its decomposition

$$\Psi=\sum_{\mu\in P_\ell}\Psi_\mu,$$

where $\Psi_\mu\in V_\mu^\dagger$. Let $\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}}=\mathcal{H}_\mu\otimes\mathcal{H}_{\mu^\dagger}\otimes\mathcal{H}_{\vec{\lambda}}$. First we construct an element $\widetilde{\Psi}_\mu\in\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}},\mathbb{C})$ and then show that it obeys the gauge condition. The $\mathfrak{g}$-module $V_\mu\otimes V_{\mu^\dagger}$ is generated by the elements of the form,

$$\rho_{P'}(X_1).\rho_{P'}(X_2)\cdots\rho_{P'}(X_n)\rho_{P''}(Y_1)\rho_{P''}(Y_2)\cdots\rho_{P''}(Y_m).0_{\mu,\mu^\dagger},$$

where X_i and Y_i are elements in the Lie algebra. For $v\in\mathcal{H}_{\vec{\lambda}}$ we define

$$\langle\widetilde{\Psi}_\mu,0_{\mu,\mu^\dagger}\otimes v\rangle=\langle\Psi,v\rangle.$$

Now we know that $\widehat{\mathcal{U}}(\Psi_\mu)\subset\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}},\mathbb{C})$. Since the diagonal $\mathfrak{g}$ action on $0_{\mu,\mu^\dagger}$ is trivial we have,

$$\rho_{P'}(Y_i)0_{\mu.\mu^\dagger}+\rho_{P''}(Y_i)0_{\mu,\mu^\dagger}=0.$$

Thus,

$$\begin{aligned}&\rho_{P''}(Y_1)\rho_{P''}(Y_2)\cdots\rho_{P''}(Y_m).0_{\mu,\mu^\dagger}\otimes v\\&\quad=(-1)\rho_{P''}(Y_1)\rho_{P''}(Y_2)\cdots\rho_{P'}(Y_m).0_{\mu,\mu^\dagger}\otimes v\\&\quad=(-1)\rho_{P'}(Y_m)\rho_{P''}(Y_1)\rho_{P''}(Y_2)\cdots\rho_{P''}(Y_{m-1}).0_{\mu,\mu^\dagger}\otimes v\\&\quad=(-1)^m\rho_{P'}(Y_m)\rho_{P'}(Y_2)\cdots\rho_{P'}(Y_1).0_{\mu,\mu^\dagger}\otimes v.\end{aligned}$$

We define

$$\begin{aligned}&\langle\widetilde{\Psi}_\mu,\rho_{P'}(X_1).\rho_{P'}(X_2)\cdots\rho_{P'}(X_n)\rho_{P''}(Y_1)\rho_{P''}(Y_2)\cdots\rho_{P''}(Y_m).0_{\mu,\mu^\dagger}\otimes v\rangle\\&\quad=(-1)^m(\Psi_\mu,\rho_{P'}(X_1(0)).\rho_{P'}(X_2(0))\cdots\rho_{P'}(X_n(0))\rho_{P'}(Y_m(0))\cdots\rho_{P'}(Y_1(0)))(v).\end{aligned}$$

The above is well defined and we have an element,

$$\widetilde{\Psi}_\mu \in \mathrm{Hom}_{\mathbb{C}}(V_\mu \otimes V_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbb{C}).$$

This is the base case. Now we assume that for nonnegative integers p and q we have produced an element $\widetilde{\Psi}_\mu \in \mathrm{Hom}_{\mathbb{C}}(F_p\mathcal{H}_\mu \otimes F_q\mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbb{C})$. We would like to define an element $\widetilde{\Psi}_\mu \in \mathrm{Hom}_{\mathbb{C}}(F_{p+1}\mathcal{H}_\mu \otimes F_q\mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbb{C})$. We note that any element of $F_{p+1}\mathcal{H}_\mu$ can be written as $X(n)u$; where n is a integer and $u \in F_p\mathcal{H}_\mu$. Thus we consider the element

$$\rho_{P'}(X(n)).u \otimes u' \otimes v \in F_{p+1}\mathcal{H}_\mu \otimes F_q\mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}},$$

where $u' \in F_q\mathcal{H}_{\mu^\dagger}$.

Now we choose a meromorphic function $\widetilde{f}$ on the curve $\widetilde{C}$ such that

$$\begin{aligned} \widetilde{f} &\in H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(*\sum_{j=1}^N Q_j + *P' + *P'')), \\ \widetilde{f} &= \xi'^m \; mod \; (\xi'^M) \; at \; P', \\ \widetilde{f} &= 0 \; mod \; (\xi''^M) \; at \; P''; \end{aligned}$$

where M is a positive integer large enough such that $\rho_{P'}(X(n)).u = 0$ and $\rho_{P''}(X(n)).u = 0$ for all $n \geq M$. The integer M (as we have remarked before) depends on the element $X \in \mathfrak{g}$. Now, let us define

$$\langle \widetilde{\Psi}_\mu, \rho_{P'}(X(n)).u \otimes u' \otimes v \rangle = -\sum_{j=1}^N \langle \widetilde{\Psi}_\mu, \rho_j(X[f]).u \otimes u' \otimes v \rangle.$$

The above is well defined, i.e. independent of the function $\widetilde{f}$ chosen. If there is another function $\widetilde{g}$ satisfying the same properties as $\widetilde{f}$, then we consider the function $\widetilde{f} - \widetilde{g}$. On the curve C we can consider $\widetilde{f} - \widetilde{g}$ as element of $H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j))$. Thus by gauge symmetry we have,

$$\sum_{j=1}^N \langle \widetilde{\Psi}_\mu, \rho_j(X[\widetilde{f}]).u \otimes u' \otimes v \rangle = \sum_{j=1}^N \langle \widetilde{\Psi}_\mu, \rho_j(X[\widetilde{g}]).u \otimes u' \otimes v \rangle.$$

We have constructed an element $\widetilde{\Psi}_\mu \in \mathrm{Hom}_{\mathbb{C}}(F_{p+1}\mathcal{H}_\mu \otimes F_q\mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbb{C})$. Similarly we can construct an element

$$\widetilde{\Psi}_\mu \in \mathrm{Hom}_{\mathbb{C}}(F_p\mathcal{H}_\mu \otimes F_{q+1}\mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbb{C}).$$

Continuing in this way we can show that there is an element

$$\widetilde{\Psi}_\mu \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_\mu \otimes \mathcal{H}_{\mu^\dagger} \otimes \mathcal{H}_{\vec{\lambda}}, \mathbb{C}).$$

The construction of the element $\widetilde{\Psi}_\mu$ using the function $\widetilde{f}$ at each stage ensure that $\widetilde{\Psi}_\mu$ obeys the gauge condition. Thus

$$\widetilde{\Psi}_\mu \in \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{X}})$$

and $\widehat{\iota}_\mu(\widetilde{\Psi}_\mu) = \Psi_\mu$. Thus $\widehat{\iota}$ is surjective. □

Combining the Last three lemmas, we see the map $\widehat{\iota}$ defines a canonical $\mathbb{C}$-linear isomorphism of $\mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{X}})$ on $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$. This completes the proof of the main theorem in this section. The following is immediate.

Corollary 4. *There is a canonical isomorphism*

$$\mathcal{V}_{\mu,\mu^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{X}}) \simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}).$$

Remark: The above theorem also known as the *Factorization theorem* is very significant step as it enable us to compute the dimension of conformal block on pointed stable curves of genus g via an induction on the genus of the curve. We will show that the dimension of the conformal block only depends upon the genus of the curve and the representations. To compute the dimension of a conformal block on a smooth N-pointed curve C of genus g we deform the curve to a N-pointed stable curve $\widehat{C}$ of genus g. We consider the normalization $\widetilde{C}$ of the curve $\widehat{C}$. The genus of $\widetilde{C}$ is less than the genus of the curve $\widehat{C}$ unless $\widehat{C}$ is a pointed projective line. Using the normalization theorem we see that we are reduced to compute the dimension of conformal blocks on a curve of genus less than g. We continue this step till we are reduced to the case of the pointed projective line. For projective space everything can be explicitly calculated. In the later sections we describe all these in more details.

For N-pointed projective line we have proved earlier

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}) \subset \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C}),$$

where $\mathfrak{X}$ is the data associated to the N-pointed $\mathbb{P}^1$. But this is not true if the genus of the underlying curve is positive. Take for example

$$\mathfrak{X} = (E; Q; \eta);$$

where E is an elliptic curve, Q is a point η is a formal neighborhood. For the trivial representation at the point Q, it can be shown that

$$\dim_{\mathbb{C}} \mathcal{V}^{\dagger}_{0}(\mathfrak{X}) = \ell + 1,$$

where as $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(V_o, \mathbb{C}) = 1$.

Chapter 6

Kodaira-Spencer Mapping

In these section we mostly give definitions and state without proof some theorems from complex deformation theory and algebraic geometry. We will be using them in the later chapters while proving the local freeness of the *sheaf of conformal blocks*.

6.1 Complex analytic family

We start with a *complex analytic family of compact Riemann surfaces*. It means we have a holomorphic mapping π between two complex manifolds $\pi : \mathcal{C} \to \mathcal{B}$; where $\mathcal{C}$ is a complex $(m+1)$ dimensional manifold and $\mathcal{B}$ is a complex m dimensional manifold which satisfies the following conditions:

(1) π is proper,

(2) π is smooth and holomorphic and for any point $P \in \mathcal{C}$, the derivative at P

$$d\pi_P : T_P(\mathcal{C}) \to T_{f(P)}\mathcal{B}$$

is surjective,

(3) For any point $b \in \mathcal{B}$ the fiber $\pi^{-1}(b)$ is connected.

The conditions imply that for every point $b \in \mathcal{B}$, the fiber at C_b at b is a compact Riemann surface. Also for a point b_0 in $\mathcal{B}$, the compact Riemann surface C_b is a deformation of the Riemann surface C_{b_0}. Given a complex analytic family of compact Riemann surfaces and a holomorphic map $h : \mathcal{S} \to \mathcal{B}$, we define the pullback of the family $\pi : \mathcal{C} \to \mathcal{B}$ to be the fiber product $\mathcal{C} \times_{\mathcal{B}} \mathcal{S} \to \mathcal{S}$. It can be easily shown that the pull back family of a complex analytic family is also complex analytic.

$$\begin{array}{ccc} \mathcal{C} \times_{\mathcal{B}} \mathcal{S} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \pi \\ \mathcal{S} & \xrightarrow{h} & \mathcal{B} \end{array}$$

We wish to study a complex analytic family locally $\pi : \mathcal{C} \to \mathcal{B}$ by using local coordinates. Choose a point 0 in the base $\mathcal{B}$ and a coordinate neighborhood U centered at 0. Let

$u = (u_1, u_2, \cdots, u_m)$ be the local coordinates. Now Let $\pi^{-1}(U)$'s is covered by open sets U_λ such that

$$\pi^{-1}(U) = \bigcup_{\lambda \in I} U_\lambda.$$

As we vary the parameter u in the base we change the patching of $\pi^{-1}(U)$. Further we can assume that U_λ's are coordinate neighborhood with coordinates (u, z_λ). For $u = 0$, we get a complex structure for the Riemann surface C_0. If we change u a little from 0, we get a complex structure of the Riemann surface C_b. Now when $u = 0$, the equation connecting the z_λ's looks like,

$$z_\lambda = f_{\lambda,\mu}(0, z_\mu).$$

Similarly for any other point $a = (a_1, a_2, \cdots, a_m)$, the compact Riemann surface C_a has a patching functions of the form,

$$z_\lambda = f_{\lambda,\mu}(a, z_\mu).$$

The degree one term in the Taylor expansion of $f_{\lambda,\mu}$ at point 0 gives the first order deformation of the complex structures of C_0. More concretely if $U_\lambda \cap U_\mu \neq \emptyset$, then

$$\{\frac{\partial f_{\lambda,\mu}}{\partial u^k}(0, z_\mu)\} \quad 1 \leq k \leq m$$

contains information about deformations of the compact Riemann surface C_0.

For $U_\lambda \cap U_\mu \neq \emptyset$, we consider the holomorphic tangent vector field

$$\theta^{(k)}_{\lambda,\,u} = \frac{\partial f_{\lambda,\mu}}{\partial u^k}(0, z_\mu)\frac{\partial}{\partial z_\lambda}$$

on the curve $C_o \cap U_\lambda \cap U_\mu$. For a triple intersection $U_\lambda \cap U_\mu \cap U_\beta \neq \emptyset$ we have,

$$\begin{aligned} z_\lambda &= f_{\lambda,\mu}(u, z_\mu) \\ &= f_{\lambda,\mu}(u, f_{\mu,\beta}(u, z_\beta)) \\ &= f_{\lambda,\beta}(u, z_\beta). \end{aligned}$$

Thus on the triple intersection $U_\lambda \cap U_\mu \cap U_\beta \neq \emptyset$,

$$\begin{aligned} \theta^{(k)}_{\lambda,\beta} &= \frac{\partial f_{\lambda,\beta}}{\partial u^k}(0, z_\beta)\frac{\partial}{\partial z_\lambda} \\ &= \frac{\partial f_{\lambda,\mu}}{\partial z_\mu}(0, f_{\mu,\beta}(0, z_\beta)).\frac{\partial f_{\mu,\beta}}{\partial u^k}(0, z_\beta)\frac{\partial}{\partial z_\lambda} \\ &\quad + \frac{\partial f_{\lambda,\mu}}{\partial u^k}(0, f_{\lambda,\mu}(f_{\mu,\beta}(0, z_\beta))\frac{\partial}{\partial z_\lambda} \\ &= \frac{\partial f_{\mu,\beta}}{\partial u^k}(0, z_\beta)(\frac{\partial f_{\lambda,\mu}}{\partial z_\mu}(0, f_{\mu,\beta}(0, z_\beta))\frac{\partial}{\partial z_\lambda}) + \theta^{(k)}_{\lambda,\mu} \\ &= \frac{\partial f_{\mu,\beta}}{\partial u^k}(0, z_\beta)\frac{\partial}{\partial z_\mu} + \theta^{(k)}_{\lambda,\mu} \\ &= \theta^{(k)}_{\mu,\beta} + \theta^{(k)}_{\lambda,\mu}. \end{aligned}$$

We see that $\{\theta^{(k)}_{\lambda,\mu}\}$ is a *Čech* 1-cocycle with coefficients in the holomorphic tangent vector fields. Hence it defines a cohomology class in $H^1(C_0,\Theta)$; where Θ is the sheaf of germs of holomorphic vector fields on C_0. The cohomology class of $\{\theta^{(k)}_{\lambda,\mu}\}$ is uniquely determined if we fix local coordinates u and is independent of the cover $\pi^{-1}(U)$ by coordinate neighborhoods with local coordinate (u, z_λ). We can define a linear mapping ρ_0 from the holomorphic tangent vector space at 0 of $\mathcal{B}$ to $H^1(C_0,\Theta)$ by

$$\rho_0 : T_0\mathcal{B} \quad \rightarrow \quad H^1(C_0,\Theta), \tag{6.1}$$

$$\sum a_k \frac{\partial}{\partial w^k} \quad \rightarrow \quad \sum a_k\{\theta^{(k)}_{\lambda,\mu}.\} \tag{6.2}$$

This linear mapping is known as the *Kodaira Spencer Mapping*. There is nothing special about the point 0 in $\mathcal{B}$, in particular if we put

$$\theta^{(k)}_{\lambda,\mu}(w) = \frac{\partial f_{\lambda,\mu}}{\partial w^k}(z_\mu, w)\frac{\partial}{\partial z_\lambda},$$

by the same argument we can show that $\{\theta^{(k)}_{\lambda,\mu}(w)\} \in H^1(C_w, \Theta_{C_w})$. As before we have a $\mathcal{O}_{\mathcal{B}}$-module map

$$\rho : \Theta_{\mathcal{B}} \quad \rightarrow \quad R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}},$$

$$\sum a_k(w)\frac{\partial}{\partial w^k} \quad \rightarrow \quad \sum a_k(w)\{\theta^{(k)}_{\lambda,\mu}(w)\}.$$

The mapping ρ is a Sheaf Version of the mapping defined in 6.1. $\Theta_{\mathcal{C}/\mathcal{B}}$ is the sheaf of relative holomorphic tangent vector fields of $\pi : \mathcal{C} \rightarrow \mathcal{B}$ which is also defined by the exact sequence

$$0 \rightarrow \Theta_{\mathcal{C}/\mathcal{B}} \rightarrow \Theta_{\mathcal{C}} \rightarrow \pi^*\Theta_{\mathcal{B}} \rightarrow 0.$$

The mapping ρ in this case is also called the Kodaira-Spencer mapping. The Kodaira-Spencer map is the connecting homomorphism τ as shown below:

$$\pi_*\Theta_{\mathcal{C}} \rightarrow \Theta_{\mathcal{B}} \rightarrow^{\tau} R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}} \rightarrow R^1\pi_*\Theta_{\mathcal{C}}.$$

6.1.1 Versal family

A complex analytic family of compact Riemann surfaces $\pi : \mathcal{C} \rightarrow \mathcal{B}$ is said to be complete at a point $0 \in \mathcal{B}$ if it satisfies the following properties:

(1) Let $C_0 = \pi^{-1}(0)$. Given a complex analytic family $\widetilde{\pi} : \mathcal{M} \rightarrow \mathcal{S}$ of compact Riemann surfaces, a point $s \in \mathcal{S}$ and an analytic isomorphism $f_0 : \widetilde{\pi}^{-1}(s) \rightarrow C_0$ there exists a holomorphic mapping g from a neighborhood U of s in $\mathcal{S}$ to $\mathcal{B}$ and a map f from $\pi^{-1}(U)$ to $\mathcal{C}$ such that

$$g(s) = 0$$

$$f_{|\widetilde{\pi}^{-1}(s)} = f_0.$$

More over for any point $a \in U$, the map f restricted to $\pi^{-1}(u)$ is an analytic isomorphism from $\widetilde{\pi}^{-1}(a)$ to $\pi^{-1}(g(a))$ with the commutative diagram

$$\begin{array}{ccc} \widetilde{\pi}^{-1}(U) & \xrightarrow{f} & \mathcal{C} \\ \downarrow & & \downarrow \\ U & \xrightarrow{g} & \mathcal{B}. \end{array}$$

In this case $\widetilde{\pi}^{-1}(U) \to U$ is complex analytically isomorphic to the pull back of the family $\pi : \mathcal{C} \to \mathcal{B}$ to U by the map g.

If the family $\pi : \mathcal{C} \to \mathcal{B}$ also satisfies the condition that dg_s is uniquely determined, the family $\pi : \mathcal{C} \to \mathcal{B}$ is said to be *versal* at the point 0. If g is uniquely determined, then the family $\pi : \mathcal{C} \to \mathcal{B}$ is defined to be *universal* at the point 0.

We state the following theorems without proof which are useful for applications.

Theorem 14. *If a complex family $\pi : \mathcal{C} \to \mathcal{B}$ of compact Riemann surfaces of genus $g \geq 2$ is complete at every point of a neighborhood of $0 \in \mathcal{B}$ and versal at the point 0, then the family $\pi : \mathcal{C} \to \mathcal{B}$ is universal at the point 0.*

The next theorem connects the Kodaira-Spencer mapping with a versal Family.

Theorem 15. *If the Kodaira-Spencer mapping of a complex analytic family $\pi : \mathcal{C} \to \mathcal{B}$ is surjective at a point $b \in \mathcal{B}$, then the family is complete at the point b.*

Theorem 16. *A complex analytic family $\pi : \mathcal{C} \to \mathcal{B}$ is versal at the point $0 \in \mathcal{B}$ if and only if the family $\pi : \mathcal{C} \to \mathcal{B}$ is complete at each point in a neighborhood of the point $0 \in \mathcal{B}$ and the Kodaira-Spencer mapping $\rho_0 : T_0\mathcal{B} \to H^1(C_0, \Theta)$ at the point 0 is an isomorphism.*

For a proof of these theorems we refer the reader to [U1]

6.2 Deformation of pointed curves

For a compact Riemann surface we have discussed the deformation in the previous section. We have seen that the data of deformation is captured in the Kodaira-Spencer mapping and are parameterized by the cohomology group $H^1(C, \Theta_C)$. Similarly the deformation of the data

$$\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \cdots, Q_N; \eta_1^{(n)}, \eta_2^{(n)}, \cdots, \eta_N^{(n)})$$

of a N-pointed Riemann surface of genus g with n-th infinitesimal neighborhood is parameterized by the group $H^1(C, \Theta_C(-(n+1)\sum_{j=1}^{N} Q_j)))$. Now we would like to do the same for semistable N-pointed curves.

If C is a nodal curve, a deformation of C is defined as a proper flat holomorphic mapping $\pi : X \to Y$ of complex spaces with a prescribed point $y \in Y$ such that $\pi^{-1}(y)$ is isomorphic to the curve C. In this case, the infinitesimal deformation is parameterized by the cohomology group $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$. The infinitesimal deformations of a stable N-pointed curve

$$\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \cdots, Q_N; \eta_1^{(n)}, \eta_2^{(n)}, \cdots, \eta_N^{(n)})$$

with n-th infinitesimal neighborhoods are parameterized by the cohomology group

$$\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C(-(n+1)\sum_{j=1}^{N} Q_j));$$

where Ω^1_C is the sheaf of *Kahler differential* on the curve C. We can regard the exact sequence,

$$0 \to \mathcal{I}_C/\mathcal{I}_C^2 \to \Omega^1_X \otimes \mathcal{O}_{\mathcal{C}} \to \Omega^1_C \to 0$$

as the definition of the sheaf Ω^1_C; where $\mathcal{I}_C$ is the ideal sheaf of the curve C in X. Let $\Theta_C = \underline{\mathrm{Hom}}_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$. We have an exact sequence of the form,

$$\begin{aligned} 0 \to H^1(C, \Theta_C(-(n+1)\sum_{j=1}^{N} Q_j)) &\to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C(-(n+1)\sum_{j=1}^{N} Q_j)) \\ &\to H^0(C, \underline{\mathrm{Ext}}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)) \to 0. \end{aligned}$$

The group $H^1(C, \Theta_C(-(n+1)\sum_{j=1}^{N} Q_j))$ corresponds to an infinitesimal deformation of the data

$$\mathfrak{X}^{(n)} = (C; Q_1, Q_2, \cdots, Q_N; \eta_1^{(n)}, \eta_2^{(n)}, \cdots, \eta_N^{(n)})$$

preserving the singularities.

Definition The data $(\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N; \eta_1, \eta_2, \cdots, \eta_N)$ is called a holomorphic family of N-pointed stable curves of genus g with formal neighborhood if the following conditions are satisfied:

(1) $\mathcal{C}$ and $\mathcal{B}$ are connected smooth complex manifolds, $\pi : \mathcal{C} \to \mathcal{B}$ is a proper flat holomorphic map and $s_1, s_2, \cdots, s_N$ are holomorphic sections of π.

(2) For each point $b \in \mathcal{B}$ the data $(C_b = \pi^{-1}; s_1(b), s_2(b), \cdots, s_N(b))$ is an N-pointed stable curve of genus g.

(3) η_j is an $\mathcal{O}_{\mathcal{B}}$-algebra isomorphism,

$$\eta_j : \mathcal{O}_{\mathcal{C}}/s_j(\mathcal{B}) := \lim_{n\to\infty} \mathcal{O}_{\mathcal{C}}/I_j^n \simeq \mathcal{O}_{\mathcal{B}}[[\xi]];$$

where I_j is the defining ideal of $s_j(\mathcal{B})$ in $\mathcal{C}$.

Similarly we can define a family of stable N-pointed curves of genus g

$$(\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N; \eta_1^{(n)}, \eta_2^{(n)}, \cdots, \eta_N^{(n)})$$

by changing the third condition by the $\mathcal{O}_{\mathcal{B}}$-algebra isomorphism:

$$\eta_j^{(n)} : \mathcal{O}_{\mathcal{C}}/I_j^{n+1} \simeq \mathcal{O}_{\mathcal{B}}[[\xi]]/I_j^{(n+1)}.$$

For a holomorphic mapping $f : \mathcal{S} \to \mathcal{B}$ we can define the pull back $f^*(\mathfrak{X}^{(n)})$ of the family of the N-pointed stable curves with n-th infinitesimal neighborhoods over the base $\mathcal{S}$ by the mapping f.

Definition A family

$$\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N; \eta_1^{(n)}, \eta_2^{(n)}, \cdots, \eta_N^{(n)})$$

of N-pointed curves with n-th formal neighborhoods is said to be *versal* (resp *universal*) at a point $b \in \mathcal{B}$, if for any deformation

$$\widetilde{\pi} : \mathcal{M} \to \mathcal{S}$$

of the data $\pi^{-1}(b) = (C_b, Q_1, Q_2, \cdots, Q_N; \eta_1, \eta_2, \cdots, \eta_N)$ with a prescribed point $s \in \mathcal{S}$, there exists a holomorphic mapping (unique holomorphic mapping) from a neighborhood of $s \in \mathcal{S}$ to $\mathcal{B}$ such that the pullback family $f^*\mathfrak{X}$ is isomorphic to the family $\widetilde{\pi}$ in a neighborhood of s and furthermore df(respectively f) is uniquely determined at the point s. If the family is *versal* at all points of the base $\mathcal{B}$, then the family $\mathfrak{X}$ is called *versal* (*universal*). As in the previous section we state some theorems relating the Kodaira-Spencer mapping with a versal family.

For a proof we refer to [U2].

Theorem 17. *For a family $(\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N)$ of N-pointed stable curves of genus g and for each point $b \in \mathcal{B}$, there exists a $\mathbb{C}$-linear mapping*

$$\rho_b : T_b\mathcal{B} \to Ext^1_{\mathcal{O}_{C_b}}(\Omega^1_{C_b}, \mathcal{O}_{C_b}(-\sum_{j=1}^{N} s_j(b)));$$

where $C_b = \pi^{-1}(b)$.

This $\mathbb{C}$-linear mapping ρ_b is called the Kodaira-Spencer mapping of the family

$$(\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N)$$

at the point b. A sheaf version of the above theorem is given by the following corollary.

Corollary 5. *With the same setting as the previous theorem, the Kodaira-Spencer mapping ρ_b at a point $b \in \mathcal{B}$ induces an $\mathcal{O}_{\mathcal{B}}$-module homomorphism*

$$\rho : \Theta_{\mathcal{B}} \to R^1_{\pi *}\underline{Hom}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_{\mathcal{C}}(-(n+1)\sum_{j=1}^{N} s_j(\mathcal{B}))).$$

The criterion of the versality as in the case of the complex analytic family of compact Riemann surfaces is given by,

Proposition 8. *A family $(\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N)$ of N-pointed stable curves of genus g is versal a point $b \in \mathcal{B}$ if and only if the Kodaira-Spencer mapping*

$$\rho_b : T_b\mathcal{B} \to Ext^1_{\mathcal{O}_{C_b}}(\Omega^1_{C_b}, \mathcal{O}_{C_b}(-\sum_{j=1}^{N} s_j(b)))$$

is an isomorphism at the point b; where $C_b = \pi^{-1}(b)$.

The existence of versal family of N-pointed stable curves is given by the following theorem.

Theorem 18. *For each N-pointed stable curves $(C, Q_i, Q_2, \cdots, Q_N)$ of genus g there exists a family*

$$\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N)$$

with prescribed point $b \in \mathcal{B}$, such that $\pi^{-1}(b)$ is isomorphic to $(C, Q_1, Q_2, \cdots, Q_N)$. More over the family $\mathfrak{F}$ is versal at the point b . Further if $\mathcal{C}$ and $\mathcal{B}$ are complex manifolds, then the family $\mathfrak{F}$ is versal at each point of a small neighborhood of $b \in \mathcal{B}$. If the automorphism group of the N-pointed stable curve is trivial then the family $\mathfrak{F}$ is also universal at b.

For a versal family

$$\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N)$$

of N-pointed stable curves of genus g. We define

$$\begin{aligned} \Sigma &= \{P \in \mathcal{C} | d\pi_P : T_P\mathcal{C} \to T_{\pi(P)}\mathcal{B} \text{ is not surjective}\} \\ D &= \pi(\Sigma). \end{aligned}$$

The set Σ is called the critical locus of the family $\mathfrak{F}$ and the set D is called the discriminant locus of the family $\mathfrak{F}$. We have the following theorem.

Theorem 19. *Assuming $2g - 2 + N \geq 1$. For a versal family $\mathfrak{F}$ of N-pointed stable curves of genus g,*

$$\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N).$$

(1) We have

$$\begin{aligned} \dim \mathcal{B} &= 3g - 3 + N \quad \textit{and} \\ \dim \mathcal{C} &= 3g - 2 + N, \end{aligned}$$

(2) The critical locus Σ is a smooth subvariety of codimension 2 in $\mathcal{C}$,

(3) The discriminant locus D is divisor with normal crossings in $\mathcal{B}$.

For a proof we refer to [Ar], [U1].

6.3 Versal family of stable pointed curves

We start with a versal family

$$\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N)$$

of N-pointed stable curves of genus g. We study the family locally. For that purpose we introduce the following local coordinates of $\mathcal{C}$. For a point P in Σ of the critical locus of π we can choose a neighborhood $(u_1, u_2, \cdots, u_M, z, w)$ of $\mathcal{C}$ with center P. The local coordinates $(\tau_1, \tau_2, \cdots, \tau_M)$ of $\mathcal{B}$ with center $\pi(P)$ such that the map is given by,

$$(u_1, u_2, \cdots, u_{M-1}, z, w) \longrightarrow (u_1, u_2, \cdots, u_{M-1}, z.w) = (\tau_1, \tau_2, \cdots, \tau_M).$$

For a point $P \in \mathcal{C} - \Sigma$, we can choose local coordinates $(u_1, u_2, \cdots, u_M, z)$ centered at P of $\mathcal{C}$ such the mapping π is given by,

$$(u_1, u_2, \cdots, u_M, z) \longrightarrow (u_1, u_2, \cdots, u_M) = (\tau_1, \tau_2, \cdots, \tau_M).$$

Since the fibers of the family may not be smooth we ca not talk about the sheaf of relative tangent vector fields. Instead we start with the relative sheaf of differentials ${\Omega^1}_{\mathcal{C}/\mathcal{B}}$ which is defined by the exact sequence

$$\pi^{-1}\Omega^1_{\mathcal{C}/\mathcal{B}} \otimes_{\mathcal{O}_\mathcal{B}} \to {\Omega^1}_{\mathcal{C}} \to {\Omega^1}_{\mathcal{C}/\mathcal{B}} \to 0.$$

The sheaf ${\Omega^1}_{\mathcal{C}/\mathcal{B}}$ is called the sheaf of germs of relative 1-forms of the family $\pi : \mathcal{C} \to \mathcal{B}$. Locally in a neighborhood of a point $P \in \mathcal{C} \setminus \Sigma$, it is isomorphic to $\mathcal{O}_\mathcal{C} dz$ and around a point $P \in \Sigma$ we have an $\mathcal{O}_\mathcal{C}$-module isomorphism

$$\Omega^1_{\mathcal{C}/\mathcal{B}} \simeq (\mathcal{O}_\mathcal{C} dz + \mathcal{O}_\mathcal{C} dw)/\mathcal{O}_\mathcal{C}(wdz + zdw).$$

We define the *relative dualizing sheaf of the family* $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N)$ to be the $\mathcal{O}_\mathcal{C}$-module

$$\omega_{\mathcal{C}/\mathcal{B}} = \omega_\mathcal{C} \otimes (\pi^* \omega_\mathcal{B}^{-1});$$

where $\omega_\mathcal{C}$ is the canonical sheaf of a complex manifold $\mathcal{C}$.

Now we put $\Theta_{\mathcal{C}/\mathcal{B}} = \underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{C}}(\Omega^1_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_\mathcal{C})$. With this notation we have the following:

Lemma 22. $\Theta_{\mathcal{C}/\mathcal{B}}$ *defined above is an invertible* $\mathcal{O}_\mathcal{C}$*-module and there is an isomorphism*

$$\Theta_{\mathcal{C}/\mathcal{B}} \simeq \underline{Hom}(\omega_{\mathcal{C}/\mathcal{B}}, \mathcal{O}_\mathcal{C}).$$

We define

$$\Theta_\mathcal{B}(-\mathrm{log} D) = \{X \in \Theta_\mathcal{B} | X(\mathcal{I}_D) \subset \mathcal{I}_D\};$$

where $\mathcal{I}_D$ is the sheaf of defining ideal of the divisor D in $\mathcal{B}$ and $\Theta_\mathcal{B}$ is the sheaf of holomorphic vector fields on $\mathcal{B}$.

Theorem 20. $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N)$ *be a versal family of stable* N*-pointed curves of genus* g*. Then there exists an* $\mathcal{O}_\mathcal{B}$*-module isomorphism*

$$\rho : \Theta_B(-\log D) \simeq R^1\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(-S));$$

where we put $S_j = s_j(\mathcal{B})$ *and* $S = \sum_{j=1}^N S_j$*. The homomorphism* ρ *is also known as the Kodaira-Spencer mapping.*

We end this section by explicitly computing the action of the Kodaira-Spencer map on the Lie bracket of holomorphic vector fields on $\mathcal{B}$.

Proposition 9. *In the setting as that of the previous theorem, let* X, Y *be holomorphic vector fields on* $\mathcal{B}$*.* ρ *be the Kodaira-Spencer map. We have*

$$\rho([X, Y]) = [\rho(X), \rho(Y)] + X(\rho(Y)) - Y(\rho(X));$$

where we put

$$\rho(X) = \{\theta_{\lambda,\mu}\}, \quad \rho(Y) = \{\tau_{\lambda,\mu}\}$$

and $X(\rho(Y))$ *is the cocycle*

$$\tau'_{\lambda,\mu} = X(\tau_{\lambda,\mu}).$$

Proof. We assume that $U_{\lambda,\mu}$ is a sufficiently fine cover of $\mathcal{C}$. Let $u = (u_1, u_2, \cdots, u_m)$ be the coordinates of $\mathcal{B}$ and (u, z_λ) be the coordinates of U_λ. We may also assume that U_j, for $j \in I$ is a coordinate neighborhood of $s_j(\mathcal{B})$ and locally defined by the equation $z_j = 0$. Thus as before for a nonempty intersection $U_\lambda \cap U_\mu \neq \emptyset$; we have

$$z_\lambda = f_{\lambda,\mu}(u, z_\mu).$$

The vector field $X = \sum_{j=1}^{m} a_i(u)\frac{\partial}{\partial u_i}$ induces an infinitesimal transformation of the coordinates u which is of the form

$$(u_1, u_2, \cdots, u_m) \longrightarrow (u_1 + \epsilon a_1(u), \cdots, u_m + \epsilon a_m(u));$$

where $\epsilon^2 = 0$. We write for simplicity,

$$u \longrightarrow u + \epsilon X.$$

Since the family is versal, we have

$$f_{\lambda,\mu}(u + \epsilon X, z_\mu) = f_{\lambda,\mu}(u, z_\mu) + \epsilon \ell_{\lambda,\mu}(u, z_\mu).$$

Similarly for the vector field

$$Y = \sum_{i=1}^{N} b_i(u)\frac{\partial}{\partial u_i},$$

we have

$$f_{\lambda,\mu}(u + \delta Y, z_\mu) = f_{\lambda,\mu}(u, z_\mu) + \delta m_{\lambda,\mu}(u, z_\mu);$$

where $\delta^2 = 0$. Let us put

$$\begin{aligned} \theta_{\lambda,\mu} &= \ell_{\lambda,\mu}\frac{\partial}{\partial z_\mu}; \\ \tau_{\lambda,\mu} &= m_{\lambda,\mu}\frac{\partial}{\partial z_\mu}; \end{aligned}$$

First act by the vector field X and then by Y to get

$$\begin{aligned} & f_{\lambda,\mu}(u, z_\mu + \delta Y) + \epsilon \ell_{\lambda,\mu}(u + \delta Y, z_\lambda + \delta m_{\lambda,\mu}) \\ & \quad = f_{\lambda,\mu}(u, z_\mu) + \delta m_{\lambda,\mu}(u, z_\lambda) + \epsilon \ell_{\lambda,\mu}(u, z_\lambda) + \epsilon\delta(Y(\ell_{\lambda,\mu}(u, z_\lambda))) + m_{\lambda,\mu}\frac{\partial \ell_{\lambda,\mu}}{\partial z_\lambda}. \end{aligned}$$

If we act in the reverse order, we get

$$f_{\lambda,\mu}(u, z_\mu) + \epsilon \ell_{\lambda,\mu}(u, z_\lambda) + \delta m_{\lambda,\mu}(u, z_\lambda) + \epsilon\delta(X(m_{\lambda,\mu}(u, z_\lambda))) + \ell_{\lambda,\mu}\frac{\partial m_{\lambda,\mu}}{\partial z_\lambda}.$$

If we put

$$\rho([X, Y]) = \{\widetilde{\varphi}_{\lambda,\mu}\};$$

we have

$$\begin{aligned} \widetilde{\varphi}_{\lambda,\mu} &= \{\ell_{\lambda,\mu}\frac{\partial m_{\lambda,\mu}}{\partial z_\lambda} - m_{\lambda,\mu}\frac{\partial \ell_{\lambda,\mu}}{\partial z_\lambda} + X(m_{\lambda,\mu})\}\frac{\partial}{\partial z_\lambda} \\ &= [\theta_{\lambda,\mu}, \tau_{\lambda,\mu}] + \{X(\tau_{\lambda,\mu}) - Y(\ell_{\lambda,\mu})\}\frac{\partial}{\partial z_\lambda} \\ &= [\rho(X), \rho(Y)] + X(\rho(Y)) - Y(\rho(X)). \end{aligned}$$

□

Chapter 7

Sheaf of Conformal Blocks

7.1 Sheaf of conformal blocks and coherency

In this section we define the sheaf of conformal blocks of a family $\mathfrak{F}$ of stable N pointed curves with formal coordinates. Let as before $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N; \eta_1, \cdots, \eta_N)$ be a family of N-pointed stable curves of genus g; $\mathcal{C}$ and $\mathcal{B}$ are finite dimensional complex manifolds. Also assume that each fiber of the family satisfies the conditions (5) of definition in section 1.1.2. We do not assume that the family is connected.

Definition: The sheaf $\widehat{\mathfrak{g}}_N(\mathcal{B})$ of affine Lie algebra over the base $\mathcal{B}$ is defined to be a sheaf of $\mathcal{O}_{\mathcal{B}}$-module as

$$\widehat{\mathfrak{g}}_N(\mathcal{B}) = \mathfrak{g} \otimes_{\mathbb{C}} (\bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j))) \oplus \mathcal{O}_{\mathcal{B}}.c.$$

The Lie bracket us given by,

$$\begin{aligned}
&[(X_1 \otimes f_1, \cdots, X_N \otimes f_N), (Y_1 \otimes g_1, \cdots, Y_N \otimes g_N)] \\
&\quad = ([X_1, Y_1] \otimes (f_1 g_1), \cdots, [X_N, Y_N] \otimes (f_N g_N)) \\
&\qquad \oplus \left(\sum_{j=1}^{N} \langle X_j, Y_j \rangle \operatorname*{Res}_{\xi_j = 0}(g_j df_j) \right).c,
\end{aligned}$$

where c belongs to the center; X_j, Y_j belong to the simple Lie algebra $\widehat{\mathfrak{g}}$ and f_j, g_j belong to $\mathcal{O}_{\mathcal{B}}((\xi_j))$.

Let

$$S = \sum_{j=1}^{N} s_j(\mathcal{B})$$

and we consider the sheaf of functions $\mathcal{O}_{\mathcal{C}}(*S)$ on $\mathcal{C}$ which have poles of arbitrary order at S. This is also the same as

$$\mathcal{O}(*S) = \varinjlim_{k} \mathcal{O}_{\mathcal{C}}(kS).$$

Now we consider the sheaf

$$\pi_*(\mathcal{O}_\mathcal{C}(*S)) = \varinjlim_k \pi_*\mathcal{O}_\mathcal{C}(kS).$$

The sheaf of affine Lie algebras associated to the family $\mathfrak{F}$ is defined by

$$\widehat{\mathfrak{g}}(\mathfrak{F}) = \mathfrak{g} \otimes_\mathbb{C} \pi_*(\mathcal{O}_\mathcal{C}(*S)).$$

There is a local expansion of functions using the formal neighborhood η_j which gives rise to a homomorphism

$$\widetilde{t} : \pi_*(\mathcal{O}_\mathcal{C}(*S)) \to \mathcal{O}_\mathcal{B}((\xi_j)).$$

Using the above homomorphism we can regard $\widehat{\mathfrak{g}}(\mathfrak{F})$ as a Lie subalgebra of $\widehat{\mathfrak{g}}_N(\mathcal{B})$. We fix an integer ℓ and for any

$$\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_N) \in P_\ell^N,$$

we define

$$\mathcal{H}_{\vec{\lambda}}(\mathcal{B}) = \mathcal{B} \otimes \mathcal{H}_{\vec{\lambda}}$$

and similarly we define

$$\mathcal{H}^\dagger_{\vec{\lambda}}(\mathcal{B}) = \underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{B}}(\mathcal{H}_{\vec{\lambda}}(\mathcal{B}), \mathcal{O}_\mathcal{B}).$$

The pairing on the Highest Weight Integrable $\widehat{\mathfrak{g}}$-modules also induces a canonical bilinear $\mathcal{O}_\mathcal{B}$-bilinear pairing between $\mathcal{H}^\dagger_{\vec{\lambda}}(\mathcal{B})$ and $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$

$$\langle\ ,\rangle : \mathcal{H}^\dagger_{\vec{\lambda}}(\mathcal{B}) \times \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) \to \mathcal{O}_\mathcal{B},$$

such that

$$\langle \Psi.a, \Phi\rangle = \langle \Psi, a.\Phi\rangle$$

for any $a \in \widehat{\mathfrak{g}}_N(\mathcal{B})$. The action of a on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ is given by

$$\begin{aligned}((X_1 \otimes \sum_{n\in\mathbb{Z}} a_{1,n}\xi_1^n), \cdots, (X_N \otimes \sum_{n\in\mathbb{Z}} a_{N,n}\xi_N^n))(f \otimes \Phi)\\ = \sum_{n\in\mathbb{Z}}\sum_{j=1}^N (a_{j,N}).f \otimes \rho_j(X_j(n))\Phi.\end{aligned}$$

As usual the action of $\widehat{\mathfrak{g}}_N(\mathcal{B})$ on $\mathcal{H}^\dagger_{\vec{\lambda}}(\mathcal{B})$ is the dual of its action on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$.

We define the *sheaf of covacua* $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and the *sheaf of vacua* or the *sheaf of conformal blocks* $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$ respectively as follows:

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) = \mathcal{H}_{\vec{\lambda}}(\mathcal{B})/\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B}),$$

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}) = \underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{B}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}), \mathcal{O}_\mathcal{B}).$$

As in the case of a curve

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}) = \{\Psi \in \mathcal{H}^\dagger_{\vec{\lambda}}(\mathcal{B}) | \Psi.a = 0\ \forall a \in \widehat{\mathfrak{g}}(\mathfrak{F})\}.$$

The bilinear pairing $\langle\ ,\ \rangle$ defined above descends to a nondegenerate $\mathcal{O}_\mathcal{B}$ bilinear pairing

$$\langle\ ,\ \rangle : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}) \times \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \to \mathcal{O}_\mathcal{B}.$$

With the above notations and definitions we have the following lemma:

Lemma 23. *Let s be a point on the base space $\mathcal{B}$ and*

$$\mathfrak{F}_s = (\pi^{-1}(s); s_1(s), \cdots, s_N(s); \eta_1|\pi^{-1}(s), \eta_N|\pi^{-1}(s))$$

be the data restricted to the fiber at s. Let $\mathbb{C}_s = \mathcal{O}_{\mathcal{B},s}/\mathfrak{m}_s$ where $\mathfrak{m}_s$ is the maximal ideal of functions vanishing at the point s. Then we have the following isomorphisms:

$$\begin{aligned}
\mathbb{C}_s \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) &\simeq \mathcal{H}_{\vec{\lambda}}, \\
\mathbb{C}_s \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathfrak{g}}_N(\mathcal{B}) &\simeq \widehat{\mathfrak{g}}_N, \\
\widehat{\mathfrak{g}}(\mathfrak{F}) \otimes_{\mathcal{O}_\mathcal{B}} \mathbb{C}_s &\simeq \widehat{\mathfrak{g}}(\mathfrak{F}_s), \\
\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_\mathcal{B}} \mathbb{C}_s &\simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_s).
\end{aligned}$$

Proof. The first and the second isomorphism are obvious and follows directly from the definition. For the third, we need to use the fact that if m is sufficiently large, we have the base change,

$$\mathbb{C}_s \otimes_{\mathcal{O}_\mathcal{B}} \pi_*(\mathcal{O}_\mathcal{C}(mS)) \simeq H^0(C_s, \mathcal{O}_{C_s}(m\sum_{j=1}^{N} s_j(s))).$$

Since for m large enough, we have

$$H^1(C_s, \mathcal{O}_{C_s}(m\sum_{j=1}^{N} s_j(s))) = 0,$$

for each point $s \in \mathcal{B}$. For a proof we refer to [Hart] Chapter III, cor 12.9. This gives the third isomorphism. For the fourth, we consider the following commutative diagram of exact sequences.

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
\mathbb{C}_s \otimes (\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B})) & \xrightarrow{\beta} & (\widehat{\mathfrak{g}}(\mathfrak{F}) \otimes \mathbb{C}_s)(\mathbb{C}_s \otimes \mathcal{H}_{\vec{\lambda}}(\mathcal{B})) \\
\downarrow \alpha & & \downarrow \epsilon \\
\mathbb{C}_s \otimes \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) & \xrightarrow{\gamma} & \mathcal{H}_{\vec{\lambda}} \\
\downarrow & & \downarrow \\
\mathbb{C}_s \otimes \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) & \xrightarrow{\delta} & \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_s) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

We know that γ is an isomorphism and β is surjective. Since the diagram is commutative the γ induces an isomorphism between $\mathrm{Im}(\alpha)$ and $\mathrm{Im}(\epsilon)$, which forces the mapping δ to be an isomorphism. This completes the proof. □

Let $f : Y \to \mathcal{B}$ be a holomorphic mapping and $\mathfrak{F}_Y$ be the pullback of the family $\mathfrak{F}$ by the morphism f, we have the following isomorphisms:

$$\begin{aligned}
\mathcal{O}_Y \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) &\simeq \mathcal{H}_{\vec{\lambda}}(Y), \\
\mathcal{O}_Y \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathfrak{g}}_N(\mathcal{B}) &\simeq \widehat{\mathfrak{g}}_N(Y), \\
\widehat{\mathfrak{g}}(\mathfrak{F}) \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_Y &\simeq \widehat{\mathfrak{g}}(\mathfrak{F}_Y), \\
\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_Y &\simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_Y).
\end{aligned}$$

Next we prove that $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is a coherent $\mathcal{O}_\mathcal{B}$-module. We state and prove the following.

Lemma 24. *Let $\mathcal{A}$ be a Lie algebra and $\mathcal{H}$ be a $\mathcal{A}$ module of finite type i.e. $\mathcal{H} = U(\mathcal{A}).V$ where V is a finite dimensional vector space. Suppose there exists a basis e_i of $\mathcal{A}$ such that the action of e_i on $\mathcal{H}$ is locally finite. Let $\mathcal{A}_+ = \{X \in \mathcal{A} | X.V = 0\}$ and $\mathcal{K}$ be a Lie subalgebra of $\mathcal{A}$ such that $\mathcal{K} + \mathcal{A}_+$ has finite codimension in $\mathcal{A}$. Then $\mathcal{H}/\mathcal{K}\mathcal{H}$ is finite dimensional.*

Theorem 21. *$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is a coherent $\mathcal{O}_\mathcal{B}$ module.*

Proof. We apply the above lemma to our present case by choosing $\mathcal{K} = \widehat{\mathfrak{g}}(\mathfrak{F})$, $\mathcal{A} = \widehat{\mathfrak{g}}_N(\mathcal{B})$ and $\mathcal{H} = \mathcal{H}_{\vec{\lambda}}(\mathcal{B})$. Since we know that the representation $\mathcal{H}_{\vec{\lambda}}$ is locally finite which satisfies the first conditions. Now from the Riemann-Roch theorem it follows that $(\widehat{\mathfrak{g}}(\mathfrak{F}) + \widehat{\mathfrak{g}}_N(\mathcal{B})_+)$ is of finite codimension in $\widehat{\mathfrak{g}}_N(\mathcal{B})$. Using finite codimension let e_i be the elements such that

$$\mathcal{A} = (\mathcal{K} + \mathcal{A}_+) \oplus (\bigoplus_{j=1}^{N} \mathcal{O}_\mathcal{B} e_j).$$

Thus, we get

$$U(\mathcal{A}) = \sum_{m_1, m_2, \cdots, m_N} U(\mathcal{K}).e_1^{m_1} \cdots e_N^{m_N}.U(\mathcal{A}_+).$$

Now, since $\mathcal{A}_+$ acts trivially on V, we get that $U(\mathcal{A}_+).V = V$. Hence

$$\mathcal{H} = \sum_{m_1, m_2, \cdots, m_N} U(\mathcal{K}).e_1^{m_1} \cdots e_N^{m_N}.V$$

Since e_i act on V locally finitely, $\widetilde{L} = \sum_{m_1, m_2, \cdots, m_N} e_1^{m_1} \cdots e_N^{m_N}.V$ is finite dimensional. Hence $\mathcal{H} = U(\mathcal{K}).\widetilde{L}$, where $\widetilde{L}$ is finite dimensional. Now there is a surjection from

$$\widetilde{L} \to \mathcal{H}/\mathcal{K}\mathcal{H}.$$

Thus we are done. □

7.2 Local freeness for smooth case

In this section we prove that the sheaf of conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ is locally free for a smooth family $\mathfrak{F}$. We define a $\mathcal{O}_\mathcal{B}$-submodule $\mathcal{L}(\mathfrak{F})$ of $\bigoplus_{j=1}^{N} \mathcal{O}_\mathcal{B}((\xi_j^{-1}))\frac{d}{d\xi_j}$ and an action of $\mathcal{L}$ on the sheaves of vacua and covacua as first order twisted differential operators.

7.2.1 Differential operators on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$

Let $\mathfrak{F}^{(0)} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \cdots, s_N)$ be a versal family of N-pointed stable curves of genus g. Let Σ be the locus of the double point of the fibers. Let $D = \pi(\Sigma)$. Now Σ is a nonsingular submanifold of codimension 2 in $\mathcal{C}$ and D is a divisor of $\mathcal{B}$ with normal crossing. For $j = 1, \cdots, N$; the formal coordinate

$$\eta_j : \mathcal{O}_{\mathcal{C}}/s_j(\mathcal{B}) \simeq \mathcal{O}_{\mathcal{B}}[[\xi]]$$

be given. Assume for simplicity that they are holomorphic in a neighborhood of $s_j(\mathcal{B})$ and let $\xi_j = \eta_j^{-1}(\xi)$.

$$\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N; \eta_1, \cdots, \eta_N)$$

be a family of N-pointed stable curves of genus g with formal neighborhoods. We have the following:

$$0 \to \Theta_{\mathcal{C}/\mathcal{B}} \to \Theta_{\mathcal{C}} \to^{d\pi} \pi^{-1}\Theta_{\mathcal{B}} \to 0,$$

where $\Theta_{\mathcal{C}/\mathcal{B}}$ is the sheaf of vectors fields which are tangent to the fibers of the family $\mathfrak{F}$. We define a new sheaf

$$\Theta'_{\mathcal{C},\pi} = d\pi^{-1}(\pi^{-1}\Theta_{\mathcal{B}}(-\mathrm{logD})).$$

So $\Theta'_{\mathcal{C},\pi}$ is the set of vector fields tangent along Σ, whose horizontal components i.e. if we write a local expansion in terms of the coordinates of the base $\mathcal{B}$, it is constant along the fibers. In other words let locally $u = \{u_i\}$ is a system of local coordinates of $\mathcal{B}$ and z, u form a local coordinate for $\mathcal{C}$. The mapping π is given by projection to u and $\pi(\Sigma) = D$ is given by the equation,

$$u_1.u_2 \cdots u_m = 0.$$

Then $\Theta'_{\mathcal{C},\pi}$ consists of vector fields whose germs are of the form:

$$a(z,u)\frac{\partial}{\partial z} + \sum_{i=1}^{m} f_i(u)u_i\frac{\partial}{\partial u_i} + \sum_{i=m+1}^{n} g_i(u)\frac{\partial}{\partial u_i}.$$

We also have an exact sequence,

$$0 \to \Theta_{\mathcal{C}/\mathcal{B}}(-S) \to (\Theta_{\mathcal{C}/\mathcal{B}}(*S)) \to^{b} \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{C}}[\xi_j^{-1}]\frac{d}{d\xi_j} \to 0.$$

This is just the Laurent expansion around $s_j(\mathcal{B})$ using the formal neighborhood upto the zeroth order. Applying the functor π_*; we get

$$0 \to \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(*S)) \to^{b} \bigoplus_{j=1}^{N}\bigoplus_{k=0}^{m} \mathcal{O}_{\mathcal{B}}\xi_j^{-k}\frac{d}{d\xi_j} \to R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S).$$

If we choose m large enough, the Riemann-Roch theorem tell us,

$$R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(mS) = 0.$$

Thus we have the following exact sequence:

$$0 \to \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(*S)) \to^b \bigoplus_{j=1}^N \mathcal{O}_\mathcal{B}[\xi_j^{-1}]\frac{d}{d\xi_j} \to^{phi} R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S) \to 0.$$

We can also define the sheaf $\Theta'_\mathcal{C}(mS)_\pi$ as the one consisting of germs of meromorphic vector fields of the form:

$$A(z,u)\frac{\partial}{\partial z} + \sum_{i=1}^m f_i(u)u_i\frac{\partial}{\partial u_i} + \sum_{i=m+1}^n g_i(u)\frac{\partial}{\partial u_i}.$$

where A has poles of order at most m along S. Now we have a exact sequence of sheaves of Lie algebras,

$$0 \to \Theta_{\mathcal{C}/\mathcal{B}}(mS) \to \Theta'_\mathcal{C}(mS)_\pi \to^{d\pi} \pi^{-1}\Theta_\mathcal{B}(-\mathrm{logD}) \to 0.$$

Now $\Theta'_\mathcal{C}(mS)_\pi$ is also a sheaf of Lie algebras under the usual bracket operation of vector fields. The above exact sequence is also an exact sequence of sheaves of Lie algebras also. By applying π_* and choosing m large enough, we get an exact sequence of $\mathcal{O}_\mathcal{B}$-modules,

$$0 \to \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(mS) \to \pi_*\Theta'_\mathcal{C}(mS)_\pi \to^{d\pi} \Theta_\mathcal{B}(-\mathrm{logD}) \to 0.$$

Taking limits $m \to \infty$, we get

$$0 \to \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) \to \pi_*\Theta'_\mathcal{C}(*S)_\pi \to^{d\pi} \Theta_\mathcal{B}(-\mathrm{logD}) \to 0.$$

Now we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) & \longrightarrow & \pi_*\Theta'_\mathcal{C}(*S)_\pi & \xrightarrow{d\pi} & \Theta_\mathcal{B}(-\mathrm{logD}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow p & & \downarrow \rho & & \\ 0 & \longrightarrow & \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) & \longrightarrow & \oplus_{j=1}^N \mathcal{O}_\mathcal{B}[\xi_j^{-1}]\frac{d}{d\xi_j} & \longrightarrow & R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S) & \longrightarrow & 0. \end{array}$$

ρ is the Kodaira-Spencer mapping of the family $\mathfrak{F}^{(0)}$. p is given by the taking the non-positive part of $\frac{d}{d\xi_j}$ part of Laurent expansion of the vector fields in $\pi_*\Theta_\mathcal{C}(mS)_\pi$ about $s_j(\mathcal{B})$ using formal coordinates.

The Kodaira Spencer mapping is an isomorphism of $\mathcal{O}_\mathcal{B}$-modules since the family is versal. By five lemma, we conclude p is also an isomorphism. Let $\widetilde{p}$ be the Laurent expansion at $s_j(\mathcal{B})$ of $\pi_*\Theta'_\mathcal{C}(*S)_\pi$. Now p being the projection to the nonnegative part of the Laurent expansion is an isomorphism, so $\widetilde{p}$ is injective. Let us put

$$\mathcal{L}(\mathfrak{F}) = \widetilde{p}(\pi_*\Theta'_\mathcal{C}(*S)_\pi).$$

We have an exact sequence of $\mathcal{O}_\mathcal{B}$-modules:

$$0 \to \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) \to \mathcal{L}(\mathfrak{F}) \to^\theta \Theta_\mathcal{B}(-\mathrm{logD}) \to 0.$$

The Lie bracket $[\ ,\ ]_d$ of $\mathcal{L}(\mathfrak{F})$ is by using the exact sequence of $\mathcal{O}_{\mathcal{B}}$-modules. For $\vec{\ell}$ and $\vec{m}$ in $\mathcal{L}(\mathfrak{F})$ we define the bracket

$$[\vec{\ell},\vec{m}]_d = [\vec{\ell},\vec{m}]_0 + \theta(\vec{\ell})(\vec{m}) - \theta(\vec{m})(\vec{\ell});$$

where $[\ ,\ ]_0$ is the usual bracket of formal vector fields and the action of $\theta(\vec{\ell})$ on

$$\vec{m} = (m_1\frac{d}{d\xi_1},\cdots,m_N\frac{d}{d\xi_N})$$

is defined by

$$(\theta(\vec{\ell})(m_1)\frac{d}{d\xi_1},\cdots,\theta(\vec{\ell})(m_N)\frac{d}{d\xi_N}).$$

Then the exact sequence is also that of the sheaves of Lie algebras. For $\vec{\ell} \in \mathcal{L}(\mathfrak{F})$ we define an action $D(\vec{\ell})$ on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ by,

$$D(\vec{\ell})(f\otimes\Phi) = \theta(\vec{\ell})(f)\otimes\Phi - f.(\sum_{j=1}^{N}\rho_j(T[l_j])\Phi;$$

where $f \in \mathcal{O}_{\mathcal{B}}$ and $\Phi \in \mathcal{H}_{\vec{\lambda}}$ and

$$\vec{\ell} = (\underline{\ell}_1,\cdots,\underline{\ell}_N) \in \mathcal{L}(\mathfrak{F}).$$

With this definition we have the following proposition:

Proposition 10. *For $\vec{\ell} = (\underline{\ell}_1,\cdots,\underline{\ell}_N) \in \mathcal{L}(\mathfrak{F})$, the action of $D(\vec{\ell})$ on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ defined has the following properties:*

(1) Let $f \in \mathcal{O}_{\mathcal{B}}$, we have

$$D(f\vec{\ell}) = fD(\vec{\ell}),$$

(2) For $\vec{\ell}$ and $\vec{m}$ in $\mathcal{L}(\mathfrak{F})$, we have

$$[D(\vec{\ell}),D(\vec{m})] = D([\vec{\ell},\vec{m}]_d) + \frac{c_v}{12}\sum_{j=1}^{N}Res_{\xi_j=0}\left(\frac{d^3\ell_j}{d\xi_j{}^3}m_j d\xi_j\right).id,$$

(3) For $f \in \mathcal{O}_{\mathcal{B}}$ and $\Phi \in \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$, we have

$$D(\vec{\ell})(f.\Phi) = (\theta(\vec{\ell})(f)).\Phi + f.D(\vec{\ell})(\Phi).$$

$D(\vec{\ell})$ is the first order twisted differential operator if $\theta(\vec{\ell}) \neq 0$. The action on $D(\vec{\ell})$ on $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$ is defined by,

$$D(\vec{\ell})(f\otimes\Psi) = (\theta(\vec{\ell})f)\otimes\Psi + \sum_{j=1}^{N}f.\Psi.\rho_j(T[\underline{l}_j]);$$

where $f \in \mathcal{O}_{\mathcal{B}}$ and $\Psi \in \mathcal{H}^{\dagger}_{\vec{\lambda}}$.

For any $\widetilde{\Psi} \in \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$ and $\widetilde{\Phi} \in \mathcal{H}_{\vec{\lambda}}(\mathcal{B})$, we have,

$$\langle D(\vec{\ell}).\widetilde{\Psi}, \widetilde{\Phi}\rangle + \langle \widetilde{\Psi}, D(\vec{\ell}).\widetilde{\Phi}\rangle = \theta(\vec{\ell})\langle \widetilde{\Psi}, \widetilde{\Phi}\rangle.$$

This follows from the definition of the action of $D(\vec{\ell})$. If $\widetilde{\Psi} = f_1 \otimes \Psi$ and $\widetilde{\Phi} = f_2 \otimes \Phi$ we get,

$$\begin{aligned}
&\langle D(\vec{\ell})(f_1 \otimes \Psi), f_2 \otimes \Phi\rangle + \langle f_1 \otimes \Psi, D(\vec{\ell})(f_2 \otimes \Phi)\rangle \\
&= (\theta(\vec{\ell})f_1).f_2\langle \Psi, \Phi\rangle + \sum_{j=1}^{N} f_1.f_2.\langle \Psi.\rho_j(T[\underline{l}_j]), \Phi\rangle \\
&\qquad + f_1.(\theta(\vec{\ell})f_2).\langle \Psi, \Phi\rangle - \sum_{j=1}^{N} f_1.f_2.\langle \Psi, \rho_j(T[\underline{l}_j]).\Phi\rangle \\
&= (\theta(\vec{\ell})f_1).f_2\langle \Psi, \Phi\rangle + f_1.(\theta(\vec{\ell})f_2)\langle \Psi, \Phi\rangle \\
&= (\theta(\vec{\ell})(f_1.f_2)\langle \Psi, \Phi\rangle \\
&= \theta(\vec{\ell})\langle \widetilde{\Psi}, \widetilde{\Phi}\rangle.
\end{aligned}$$

Proposition 11.

$$D(\vec{\ell})(\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B})) \subset \widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B}).$$

This proves that $D(\vec{\ell})$ acts on $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ as a first order differential operator if $\theta(\vec{\ell}) \neq 0$.

Proof. Let $t_j(h)$ be the Laurent expansion of h at $S_j = s_j(\mathcal{B})$ with respect to the parameter ξ_j. Now we know that $\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ consists of linear combination of elements of the form $f \otimes (\sum_{j=1}^{N} \rho_j(X \otimes t_j(h)).\Phi)$; where $f \in \mathcal{O}_{\mathcal{B}}$, $X \in \mathfrak{g}$, $h \in \pi_*\mathcal{O}_{\mathcal{C}}(*S)$ and $\Phi \in \mathcal{H}_{\vec{\lambda}}$. We prove the following equality:

$$[D(\vec{\ell}), \sum_{j=1}^{N} \rho_j(X \otimes t_j(h))] = \sum_{j=1}^{N} \rho_j\Big(x \otimes \big(\theta(\vec{\ell})(t_j(h) + \underline{\ell}_j(t_j(h)))\big)\Big);$$

where $t_j(h)$ has coefficients in $\mathcal{O}_{\mathcal{B}}$ and $\theta(\ell)$ operates on them. Now $\theta(\vec{\ell})(t_j(h) + \underline{\ell}_j(t_j(h)))$ is the Laurent expansion at $s_j(\mathcal{B})$ of a meromorphic function $\tau(h)$. τ is the inverse image of $\vec{\ell} \in \pi_*(\Theta'_{\mathcal{C}}(*S)_{\pi})$ under the map $\widetilde{p}$. Then the proof of the proposition follows from the above equality. By part (c) of the previous proposition, it is sufficient to prove it for

$\Phi \in \mathcal{H}_{\vec{\lambda}}$.

$$\begin{aligned}
D(\vec{\ell})(\sum_{j=1}^{N}\rho_j(X\otimes t_j(h)).\Phi) &- \sum_{j=1}^{N}\rho_j(X\otimes t_j(h))(D(\vec{\ell}).\Phi) \\
&= \sum_{j=1}^{N}\Big(\rho_j(X\otimes\theta(\vec{\ell})(t_j(h))) - T[\underline{\ell}_j]\rho_j(X\otimes t_j(h))\Big).\Phi \\
&\quad + \sum_{j=1}^{N}\rho_j(X\otimes t_j(h))T[\underline{\ell}_j].\Phi \\
&= \sum_{j=1}^{N}\Big(\rho_j(X\otimes\theta(\vec{\ell}))(t_j(h)) + \rho_j(X\otimes\underline{\ell}_j(t_j(h)))\Big).\Phi \\
&= \sum_{j=1}^{N}\Big(\rho_j(X\otimes(\theta(\vec{\ell}))(t_j(h))) + \underline{\ell}_j(t_j(h))\Big).\Phi.
\end{aligned}$$

Similarly we can also show that the $D(\vec{\ell})$ acts on $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ as twisted first order differential operator. □

7.2.2 Local freeness for smooth family

The main theorem we prove is in this section is:

Theorem 22. *The $\mathcal{O}_{\mathcal{B}}$-module $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is locally free on $\mathcal{B}\setminus D$ where $D = \pi(\Sigma)$ is the singular locus consisting of points in the base space whose fiber is a singular curve.*

Proof. Let $s \in \mathcal{B}\setminus D$; where $\mathfrak{F}_s$ is the data restricted to the fiber at s. $\mathbb{C}_s = \mathcal{O}_{\mathcal{B},s}/\mathfrak{m}_s$ where $\mathfrak{m}_s$ is the maximal ideal of functions vanishing at the point s. We have the following isomorphism:

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})\otimes_{\mathcal{O}_{\mathcal{B}}}\mathbb{C}_s \simeq \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_s).$$

Let $(v_1, v_2, \cdots, v_m)$ be local holomorphic sections of $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ in a neighborhood of s such that $\{v_1(s), \cdots, v_m(s)\}$ form a basis of $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_s)$. Suppose there is a nontrivial relation

$$a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0;$$

where a_i's are holomorphic functions in a neighborhood of s. Since $v_i's$ form a basis of $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_s)$, we get $a_i(s) = 0$ for all i. Now reordering the suffixes if necessary, we may assume that there is a positive integer k such that

$$\begin{aligned}
a_1 &\in \mathfrak{m}_s^k\setminus\mathfrak{m}_s^{k+1} \\
\text{and}\quad a_i &\in \mathfrak{m}_s^l;
\end{aligned}$$

for $i = 2, 3, \cdots, m$ and $l \geq k$. We choose a_i so that k is the smallest among the relations.

Let τ be a nowhere vanishing vector field in a neighborhood of s such that $\tau(a_1) \in \mathfrak{m}_s^{k-1}$. By an exact sequence,

$$0 \to \pi_*(\Theta_{\mathcal{C}/\mathcal{B}})(*S) \to \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}[\xi_j^{-1}]\frac{d}{d\xi_j} \to R^1\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(-S) \to 0$$

discussed earlier, we see that $\theta(\vec{\ell}) = \tau$. Now we apply τ to the relation among the sections to get,

$$\sum_{i=1}^{m}(\tau(a_i) + \sum_{j=1}^{m} a_j\alpha_{ji})v_i = 0, \tag{7.1}$$

where we have,

$$D(\vec{\ell})(v_i) = \sum_{j=1}^{m} \alpha_{ij}v_j.$$

Thus the above relation is nontrivial and

$$\tau(a_1) + \sum_{j=1}^{m} a_j\alpha_{j1} \in \mathfrak{m}_s^{k-1}.$$

This is a contradiction to the choice of k. Hence $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is a locally free sheaf of $\mathcal{O}_{\mathcal{B}}$-module at s. □

For a coherent $\mathcal{O}_{\mathcal{B}}$-module $\mathcal{G}$, the locus of points over which $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is not locally free is a closed analytic subset of $\mathcal{B}$ of codimension at least 2. So we get the following:

Corollary 6. *W be the maximal subset of $\mathcal{B}$ over which $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is not locally free, then W is a analytic subset of $\mathcal{B}$ and W is properly contained in D.*

Since

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) = \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}), \mathcal{O}_{\mathcal{B}}).$$

we get,

Corollary 7. *$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ restricted to $\mathcal{B} \setminus D$ is a locally free sheaf of $\mathcal{O}_{\mathcal{B}}$-modules and for any subvariety Y of $\mathcal{B} \setminus D$ we have an $\mathcal{O}_Y$ isomorphism*

$$\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) \simeq \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_{|Y}).$$

Let $\mathfrak{F}$ is a family of N-pointed smooth curves with formal neighborhoods. The previous discussions tell us, $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ are locally free $\mathcal{O}_{\mathcal{B}}$-modules dual to each other.

7.3 Local freeness in general case

In this section we prove the local freeness of the sheaves $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ in the general case.

7.3.1 Sewing method

Let

$$\mathfrak{F}^{(0)} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N)$$

be a versal family of N-pointed stable curves of genus g.

$$\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, s_2, \cdots, s_N; \eta_1, \cdots, \eta_N)$$

be a family over $\mathfrak{F}^{(0)}$ with formal neighborhoods. Let $D = \pi(\Sigma)$ be the discriminant locus and $D = \bigcup_{i=1}^{k} D_i$ be the decomposition of D into irreducible components. Since D is a divisor in $\mathcal{B}$ with normal crossings, the number is k is the maximal number of double points of a fiber in the family. More precisely it means that if a fiber C_s at $s \in \mathcal{B}$ has m double points, then exactly m components of D pass through s.

Let us choose D_i, consider the restriction of the family $\pi : \mathcal{C} \to \mathcal{B}$ to D_i. Denote it $\pi_{D_i} : \mathcal{C}_{D_i} \to D_i$. The normalization of $\pi_{D_i} : \mathcal{C}_{D_i} \to D_i$ along the double point of the fibers over D_i is denoted by $\widetilde{\pi}_{D_i} : \widetilde{\mathcal{C}}_{D_i} \to D_i$. Let s', s'' be the sections corresponding to the double points.

Proposition 12. *The family*

$$\mathfrak{F}^{(0)}_{D_i} = (\widetilde{\pi}_{D_i} : \widetilde{\mathcal{C}}_{D_i} \to D_i; s_1, s_2, \cdots, s_N, s', s'')$$

is a local universal family of $(N+2)$-pointed stable curves. The family may not be connected. If $\mathcal{C}_{D_i}$ is connected, then it consists of a family of stable curves of genus $g-1$. If not connected, then it consists of two disconnected families over D_i of (N_1+1) and (N_2+1)-pointed stable curves of genus g' and g'' over D_i. Also $N_1 + N_2 = N$ and $g' + g'' = g$.

Let us recall the following lemma which we have proved in chapter 2.

Lemma 25. *There exists a bilinear pairing*

$$(\ |\ ) : \mathcal{H}_\mu \times \mathcal{H}_{\mu^\dagger} \to \mathbb{C},$$

unique upto a constant multiple such that we have

$$(X(n)u|v) + (u|X(-n)v) = 0,$$

for any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$ and $u \in \mathcal{H}_\mu$ and $v \in \mathcal{H}_{\mu^\dagger}$. Moreover $(\ |\ )$ is zero on $\mathcal{H}_\mu(d) \times \mathcal{H}_{\mu^\dagger}(d')$ if $d \neq d'$.

We know that $\mathcal{H}_\mu(d)$ is a finite dimensional vector space. Let it is of dimension m_d. Choose a basis $\{v_1(d), v_2(d), \cdots, v_{m_d}(d)\}$. With respect to the pairing $(\ |\ )$, choose a dual basis $\{v^1(d), v^2(d), \cdots, v^{m_d}(d)\}$ of $\mathcal{H}_{\mu^\dagger}(d)$.

The sheaf $\mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_{D_i})$ is locally free on $D_i \backslash \cup_{j \neq i} D_j$. Consider a section Ψ of $\mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_{D_i})$. We wish to define an element $\widetilde{\Psi} \in \mathcal{H}^\dagger_{\vec{\lambda}}(D_i)[[q]]$ associated to Ψ. For each $d \geq 0$, we define an element $\Psi_d \in \mathcal{H}^\dagger_{\vec{\lambda}}(D_i)$. Then we can simply define $\widetilde{\Psi} \in \mathcal{H}^\dagger_{\vec{\lambda}}(D_i)[[q]]$ to be:

$$\widetilde{\Psi} = \sum_{d=0}^{\infty} \Psi_d q^d.$$

The action of $\widetilde{\Psi}$ on $\Phi \in \mathcal{H}_{\vec{\lambda}}(D_i)$ is given by,

$$\langle \widetilde{\Psi}, \Phi \rangle = \sum_{d=0}^{\infty} \langle \Psi_d, \Phi \rangle q^d.$$

So it remains to define the element Ψ_d. We define it by its action on any $\Phi \in \mathcal{H}^{\dagger}_{\vec{\lambda}}(D_i)$:

$$\langle \Psi_d, \Phi \rangle = \sum_{i=1}^{m_d} \langle \Psi, v_i(d) \otimes v^i(d) \otimes \Phi \rangle.$$

Definition The construction $q^{\Delta_\mu}\widetilde{\Psi}$ with the previous $\widetilde{\Psi}$ is known as *sewing*; where Δ_μ is as before defined by

$$\Delta_\mu = \frac{\langle \mu, \mu + \rho \rangle}{2(g^* + \ell)}$$

and g^* is the dual Coxeter number. It is true that the formal construction $\widetilde{\Psi}$ actually converges. We do not prove this fact here. For a proof we refer to [U1]. We show here that $\widetilde{\Psi}$ satisfies the formal gauge condition. The following two lemma makes it precise.

Lemma 26. *There is a $\mathcal{O}_{D_i}$-module injection:*

$$\begin{array}{rcl} \pi_* \mathcal{O}_{\mathcal{C}}(*S)_{|D_i} & \hookrightarrow & \widetilde{\pi}_{D_i *} \mathcal{O}_{\widetilde{\mathcal{C}}_{D_i}}(*(s' + s'' + S))[[q]] \\ f & \rightarrow & \displaystyle\sum_{k=0}^{\infty} f_k q^k; \end{array}$$

where $f_k \in \widetilde{\pi}_{D_i *} \mathcal{O}_{\widetilde{\mathcal{C}}_{D_i}}(*S + k(s' + s''))$.

Proof. The proof is by looking at the double points and analyzing the Laurent expansion at that point. Let P be a double point of a fiber of π_{D_i}. We choose local coordinates $(u_1, u_2, \cdots, u_{M-1}, z, w)$ of $\mathcal{C}$ with center P and $(u_1, u_2, \cdots, u_{M-1}, q)$ with center $\pi(P)$ of $\mathcal{B}$. π is given by projection on the first $M-1$ coordinates and on the last one it is given by the equation $q = zw$. We denote $u = (u_1, u_2, \cdots, u_{M-1})$. Since f is holomorphic at P we have an expansion of the form:

$$f(u, z, w) = \sum_{m \geq 0, n \geq 0} f_{m,n}(u) z^m w^n.$$

Let P' and P'' are the points in $\widetilde{\mathcal{C}}_{D_i}$ obtained from the point P after normalization. Let us define $g_{P'}(u, q, z)$ as:

$$g_{P'}(u, q, z) = f(u, z, \frac{q}{z}) = \sum_{k=0}^{\infty} g_k(u, z) q^k;$$

where

$$g_k(u, z) = \sum_{m=0}^{\infty} f_{m,k}(u) z^{m-k}.$$

Similarly define $h_{P''}$ as

$$h_{P''}(u,q,w) = f(u,\frac{q}{w},w) = \sum_{k=0}^{\infty} h_k(u,w)q^k;$$

where

$$h_k(u,z) = \sum_{n=0}^{\infty} f_{k,n}(u)w^{n-k}.$$

For a point Q in the fiber which is not a double point the map π is just given by projection onto the first M coordinates. Thus we have an expansion of the form:

$$f(u,q,z) = \sum_{k=0}^{\infty} f_{Q,k}(u,z)q^k.$$

Now the data $(g_k, h_k, f_{Q,k})$ defines a local holomorphic section of the sheaf

$$\widetilde{\pi}_{D_i}\mathcal{O}_{\widetilde{\mathcal{C}}_{D_i}}(*S + k(s'+s'')).$$

$\square$

Lemma 27. $\widetilde{\Psi}$ *constructed using the sewing method satisfies the formal gauge condition. In other words for* $f \in \pi_*\mathcal{O}_{\mathcal{C}}(*S)_{|D_i}$*, consider the expansion* $\sum_{k=0}^{\infty} f_k q^k$ *given by the previous lemma. Then,*

$$\sum_{j=1}^{N} \widetilde{\Psi}.\big(\sum_k \rho_j(X\otimes f_k)q^k\big) = 0.$$

Proof. We will show that for any $\Phi \in \mathcal{H}_{\vec{\lambda}}(D_i)$:

$$\langle \sum_{j=1}^{N} \widetilde{\Psi}.\big(\sum_k \rho_j(X\otimes f_k)q^k\big), \Phi\rangle = 0.$$

In the proof of the previous lemma we have,

$$\rho_{\sigma'}(X\otimes g_k) = \sum_{m=0}^{\infty} f_{m,k}(t)\rho_{\sigma'}(X(m-k)),$$

$$\rho_{\sigma'}(X\otimes h_k) = \sum_{n=0}^{\infty} f_{k,n}(t)\rho_{\sigma''}(X(n-k)).$$

Now using the above we get,

$$
\begin{aligned}
&\sum_{j=1}^{N}\langle\widetilde{\Psi}.\big(\sum_{k}\rho_j(X\otimes f_k)q^k\big),\Phi\rangle\\
&=\sum_{k=0}^{\infty}\sum_{d=0}^{\infty}\sum_{i=1}^{m_d}q^{k+d}\sum_{j=1}^{N}\langle\Psi,\rho_j(X\otimes f_k)(v_i(d)\otimes v^i(d)\otimes\Phi)\rangle\\
&=-\sum_{k=0}^{\infty}\sum_{d=0}^{\infty}\sum_{i=1}^{m_d}q^{k+d}\langle\Psi,\rho_{\sigma'}(X\otimes g_k)+\rho_{\sigma''}(X\otimes h_k).(v_i(d)\otimes v^i(d)\otimes\Phi)\rangle\\
&=-\sum_{k=0}^{\infty}\sum_{d=0}^{\infty}\sum_{i=1}^{m_d}\sum_{m=0}^{\infty}q^{k+d}f_{m,k}(t)\langle\Psi,\rho_{\sigma'}(X(m-k)).(v_i(d)\otimes v^i(d)\otimes\Phi)\rangle\\
&-\sum_{k=0}^{\infty}\sum_{d=0}^{\infty}\sum_{i=1}^{m_d}\sum_{n=0}^{\infty}q^{k+d}f_{k,n}(t)\langle\Psi,\rho_{\sigma''}(X(n-k)).v_i(d)\otimes v^i(d)\otimes\Phi)\rangle\\
&=-\sum_{k=0}^{\infty}\sum_{d=0}^{\infty}\sum_{i=1}^{m_d}\sum_{m=0}^{\infty}q^{k+d}f_{m,k}(t)\langle\Psi,\rho_{\sigma'}(X(m-k)).(v_i(d)\otimes v^i(d)\otimes\Phi)\rangle\\
-&\sum_{k=0}^{\infty}\sum_{d=0}^{\infty}\sum_{i=1}^{m_{d-m+k}}\sum_{m=0}^{\infty}q^{k+d}f_{m,k}(t)\langle\Psi,\rho_{\sigma''}(X(-m+k)).(v_i(d-m+k)\otimes v^i(d-m+k)\otimes\Phi)\rangle.
\end{aligned}
$$

So we are reduced to proving that

$$
\begin{aligned}
&\sum_{i=1}^{m_d}\rho_{\sigma'}(X(m-k)).(v_i(d)\otimes v^i(d)\otimes\Phi)+\\
&\quad\sum_{j=1}^{m_{d-m+k}}\rho_{\sigma''}(X(-m+k)).(v_i(d-m+k)\otimes v^i(d-m+k)\otimes\Phi)=0.
\end{aligned}
$$

By the bilinear form (|) described earlier we have the following equality:

$$(X(m-k)v_i(d),v^j(d-m+k))+(v_i(d)|X(k-m)v^j(d-m-k))=0.$$

The rest of the proof follows by computing the coefficients of the vectors using the pairing (|) after writing in terms of basis and using the above equality. This completes the proof. □

Let

$$\widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i}=\varinjlim_{n}\mathcal{O}_{\mathcal{B}}/I^n_{D_i}.$$

We can identify,

$$\widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i}\simeq\mathcal{O}_{D_i}[[q]].$$

Lemma 28.

$$\widehat{\mathcal{V}}^{\dagger}_{\vec{\lambda}\backslash D_i}=\{\Psi\in\mathcal{H}^{\dagger}_{\vec{\lambda}}(D_i)[[q]]\mid\sum_{j=1}^{N}\Psi.\sum_{k}\rho_j(X\otimes f_k)q^k=0\},$$

for all $f \in \pi_* \mathcal{O}_{\mathcal{C}}(*S)$. Then there is an $\mathcal{O}_{D_i}$-module isomorphism,

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i} \simeq \widehat{\mathcal{V}}^\dagger_{\vec{\lambda}\backslash D_i}.$$

Proof. Since the $\widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i}$-module is faithfully flat $\mathcal{O}_\mathcal{B}$-module, we have the following adjoint isomorphism:

$$\begin{aligned}
&\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i} \\
&\quad = \underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{B}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}), \mathcal{O}_\mathcal{B}) \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i} \\
&\quad \simeq \underline{\mathrm{Hom}}_{\widehat{\mathcal{O}}_{\mathcal{B}/D_i}}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathcal{O}}_\mathcal{B}, \widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i}).
\end{aligned}$$

We have an isomorphism,

$$\mathcal{H}_{\vec{\lambda}}(D_i)[[q]] \simeq \mathcal{H}_{\vec{\lambda}} \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i}.$$

By faithful flatness,

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i} \simeq \{\mathcal{H}_{\vec{\lambda}} \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i}\}/\{(\widehat{\mathfrak{g}}(\mathfrak{F}))\mathcal{H}_{\vec{\lambda}} \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i}\}.$$

Now,

$$\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}} \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i} \simeq \widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(D_i)[[q]].$$

Thus we have,

$$\begin{aligned}
\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i} &\simeq \{\mathcal{H}_{\vec{\lambda}} \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i}\}/\{(\widehat{\mathfrak{g}}(\mathfrak{F}))\mathcal{H}_{\vec{\lambda}} \otimes_{\mathcal{O}_\mathcal{B}} \widehat{\mathcal{O}}_{\mathcal{B}\backslash D_i}\} \\
&\simeq \mathcal{H}_{\vec{\lambda}}(D_i)[[q]]/\widehat{\mathfrak{g}}(\mathfrak{F})\mathcal{H}_{\vec{\lambda}}(D_i)[[q]].
\end{aligned}$$

□

7.3.2 Local freeness

The main theorem is,

Theorem 23. *The sheaf $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is locally free $\mathcal{O}_\mathcal{B}$-module. Hence $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$ is also locally free $\mathcal{O}_\mathcal{B}$-module. For any holomorphic mapping $Y \to \mathcal{B}$, consider the pullback $\mathfrak{F}_Y$ of the family $\mathfrak{F}$. We have a canonical isomorphism:*

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}) \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_Y \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}_Y).$$

We prove the theorem in steps. If $D = \emptyset$, then the theorem is true by 22 in this chapter. Therefore we may assume that $D \neq \emptyset$. Let $D = \cup_{i=1}^k D_i$ be the decomposition into irreducible components. We prove the theorem by induction of k. As mentioned before k is the maximal number of double points present in a fiber of the family $\mathfrak{F}^{(0)}$.

Let us assume that $k = 1$. Then the family $\widetilde{\mathfrak{F}}^{(0)}_D$ is a versal family of $(N+2)$-pointed smooth curves. Choose $\mathcal{B}$ small enough such that $\bigoplus_{\mu \in P_\ell} \mathcal{V}^\dagger_{\mu\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_D)$ is $\mathcal{O}_D$ free. Let

$$\{\Psi_1, \Psi_2, \cdots, \Psi_m\}$$

be a $\mathcal{O}_D$ basis of sections of $\mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{F}}_D)$. Using sewing construction we consider the elements

$$\{\widetilde{\Psi}_1, \widetilde{\Psi}_2, \cdots, \widetilde{\Psi}_m\}$$

The correspondence $\Psi_i \to \widetilde{\Psi}_i$, gives an $\mathcal{O}_D$-module homomorphism:

$$\imath : \bigoplus_{\mu \in P_\ell} \mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{F}}_D) \to \widehat{\mathcal{V}}^{\dagger}_{\vec{\lambda}\setminus D}.$$

Lemma 29.

$$\{\widetilde{\Psi}_1, \widetilde{\Psi}_2, \cdots, \widetilde{\Psi}_m\}$$

generate a free $\mathcal{O}_D[[q]]$-submodule of $\widehat{\mathcal{V}}^{\dagger}_{\vec{\lambda}\setminus D}$.

Proof. Suppose they have a nontrivial relation

$$\sum_{i=1}^{m} a_i(q)\widetilde{\Psi}_i = 0;$$

where $a_i(q) \in \mathcal{O}_D[[q]]$. If all the a_i vanish at 0, we can divide it by highest power of q. We can assume that not all a_i vanish at 0. Thus we put $q = 0$ in the above equation to get,

$$\sum_{i=1}^{m} a_i(0)\Psi_i = 0.$$

This is a contradiction since Ψ_i's are $\mathcal{O}_D$-linearly independent. So for all i we get, $a_i(0) = 0$. This proves $\widetilde{\Psi}_i$'s are all linearly independent hence generates a free $\mathcal{O}_D[[q]]$-submodule of $\widehat{\mathcal{V}}^{\dagger}_{\vec{\lambda}\setminus D}$. □

Now we complete the proof for the case when $k = 1$. We choose a point $x \in D$ and $s \in \mathcal{B} \setminus D$. We will show that the rank of the coherent sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ is constant. Then the conclusion will follow from Nakayama's lemma. Lemma 29 and lemma 28 tell us,

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes \mathbb{C}_s &= \dim_{\mathbb{C}} \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}) \otimes \mathbb{C}_s \\ &\geq n = \sum_{\mu \in P_\ell} \mathrm{rank} \mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{F}}_D). \end{aligned}$$

By *factorization theorem* we also have,

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes \mathbb{C}_x &= \sum_{\mu \in P_\ell} \dim_{\mathbb{C}} \mathcal{V}_{\mu,\mu^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{F}}_D) \otimes \mathbb{C}_x \\ &= \sum_{\mu \in P_\ell} \dim_{\mathbb{C}} \mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{F}}_D) \otimes \mathbb{C}_x \\ &= n. \end{aligned}$$

Hence,

$$\dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes \mathbb{C}_s \geq \dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes \mathbb{C}_x.$$

Since $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is coherent, we get the following:

$$\dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes \mathbb{C}_s \leq \dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes \mathbb{C}_x.$$

Thus we have,

$$\dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes \mathbb{C}_s = \dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes \mathbb{C}_x.$$

This proves $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is locally free.

Next we consider the case when $k \geq 2$. We have the decomposition

$$D = \cup_{i=1}^{k} D_i$$

into irreducible components. D_{sing} be the singular locus of the D. Our above argument shows that $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is locally free on $\mathcal{B} \setminus D_{sing}$.

Lemma 30. *If $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i})$ is a locally free $\mathcal{O}_{D_i}$-module for all i, then $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is a locally free $\mathcal{O}_{\mathcal{B}}$-module.*

Proof. Let

$$\operatorname{rank}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})_{|\mathcal{B}\setminus D_{sing}}) = m.$$

Then the rank of $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i})$ is also m. Hence for each point $x \in \mathcal{B}$ we have,

$$\dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_x) = \dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes \mathbb{C}_x = m.$$

This implies that $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is generated by m sections for each point $x \in \mathcal{B}$ in a neighborhood of x as an $\mathcal{O}_{\mathcal{B}}$-module. Suppose at a point $y \in D_{sing}$ the sheaf $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is not locally free, then it is generated by more than m sections as an $\mathcal{O}_{\mathcal{B}}$-module. Since $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is coherent using Nakayama's lemma we see that rank of $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F})$ is constant. Hence it is locally free. □

So our problem reduces to proving that the coherent sheaf is $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i})$ is locally free $\mathcal{O}_{D_i}$-module. Let j_{D_i} be the mapping:

$$j_{D_i} : \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i}) \to \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_{D_i})$$

by sending,

$$[\Phi] \to [0_{\mu,\mu^\dagger} \otimes \Phi];$$

where $[\Phi]$ is just the image of Φ in $\mathcal{H}_{\vec{\lambda}}(D_i)$ and $[0_{\mu,\mu^\dagger} \otimes \Phi]$ is the image of $[0_{\mu,\mu^\dagger} \otimes \Phi]$ in $\mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}}(D_i)$. With this setting we have the following lemma:

Lemma 31. *The mapping j_{D_i} is well defined and surjective and if $\mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_{D_i})$ is a locally free $\mathcal{O}_{D_i}$-module, for $\mu \in P_\ell$, then j_{D_i} is also an isomorphism. Hence, $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i})$ is locally free.*

Proof. Let Φ and Φ' be elements which are in the same equivalence class in $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i})$. So they satisfy a relation of the form:

$$\Phi' = \Phi + \sum_n (X_n \otimes f_n)\Phi_n;$$

where $(X_n \otimes f_n) \in \widehat{\mathfrak{g}}(\mathfrak{F}_{D_i})$ and $\Phi_n \in \mathcal{H}_{\vec{\lambda}}(D_i)$. Now let $\widetilde{f}_n$ be the pull back of f_n to $\widetilde{\mathfrak{F}}_{D_i}$. Since $\widetilde{f}_n$ is holomorphic at $s'(D_i)$ and $s''(D_i)$, we get the following:

$$\rho_{s'(D_i)}(X_n \otimes \widetilde{f}_n)0_{\mu,\mu^\dagger} + \rho_{s''(D_i)}(X_n \otimes \widetilde{f}_n)0_{\mu,\mu^\dagger} = 0.$$

Thus we have,

$$0_{\mu,\mu^\dagger} \otimes \Phi' = 0_{\mu,\mu^\dagger} \otimes \Phi + \sum_n (X_n \otimes \widetilde{f}_n)0_{\mu,\mu^\dagger} \otimes \Phi_n.$$

Hence,

$$[0_{\mu,\mu^\dagger} \otimes \Phi] = [0_{\mu,\mu^\dagger} \otimes \Phi'].$$

This proves that the map j_{D_i} is well defined.

Consider the exact sequence,

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i}) \to \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_{D_i}) \to \mathrm{Coker} j_{D_i} \to 0.$$

We have an isomorphism (factorization of covacua),

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i}) \otimes \mathbb{C}_x \simeq \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_{D_i}) \otimes \mathbb{C}_x.$$

Hence,

$$\mathrm{Coker} j_{D_i} \otimes \mathbb{C}_x = 0.$$

By Nakayama's lemma, $\mathrm{Coker} j_{D_i} = 0$. Thus j_{D_i} is surjective. Next we consider the sequence,

$$0 \to \mathrm{Ker} j_{D_i} \to \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i}) \to \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_{D_i}) \to 0.$$

Assume that $\bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_{D_i})$ is locally free. Then we have,

$$\underline{\mathrm{Tor}}^1_{\mathcal{O}_{D_i}}\Big(\bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_{D_i}), \mathbb{C}_x\Big) = 0.$$

Thus we have an exact sequence of the form,

$$0 \to \mathrm{Ker} j_{D_i} \otimes \mathbb{C}_x \to \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{D_i}) \otimes \mathbb{C}_x \to \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_x) \otimes \mathbb{C}_x \to 0.$$

Thus we see that,

$$\mathrm{rank}(\mathrm{Ker} j_{D_i}) = 0.$$

Hence by Nakayama's lemma, we get $\mathrm{Ker} j_{D_i} = 0$. So j_{D_i} is an isomorphism. □

After working out all these little important lemma it is time to patch them all up to give a proof of theorem 23. We prove the theorem by induction on (g, N). First for the case $g = 0$ and $N = 3$, there is only one N-pointed stable curve. So $\mathcal{B}$ is a point. Hence the theorem is true. Assume that the theorem is true for $g = 0$ and $N \leq M$. We would like to show that theorem is true for $N = M + 1$. Then for a local universal family $\mathfrak{F}^{(0)}$ of N-pointed stable curves of genus 0 and $D \neq \emptyset$, $\widetilde{\mathfrak{F}}^{(0)}_{D_i}$ is a disjoint union of local universal family of $(N' + 1)$ and $(N'' + 1)$-pointed stable curves of genus 0. Moreover $N' + N'' = N$. Since $N' + 1 \leq M$ and $N'' + 1 \leq M$, we get by induction hypothesis $\mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\widetilde{\mathfrak{F}}_{D_i})$ is locally free. Hence by lemma 31, we are done.

Now we assume that the theorem is true for any (g, N) where $g \leq h$ and $N \leq M$. Then by repeating the argument before, we see that the theorem is true for $(g, M + 1)$. Now let $\mathfrak{F}^{(0)}_{D_i}$ be a local universal family of N-pointed stable curve of genus $(h + 1)$ with $N \leq M$. If the discriminant locus of the family is empty, then the theorem is automatically true. Otherwise the discriminant locus is nonempty, then $\widetilde{\mathfrak{F}}^{(0)}_{D_i}$ is a local universal family of $(N' + 1)$-pointed stable curves of genus g' and $(N'' + 1)$ pointed stable curves of genus g'' with $N' + N'' = N$ and $g' + g'' = h$. If $g' < h$ and $g'' < h$, then by induction hypothesis the theorem is true. Otherwise if, $g' = 0$ and $N'' \geq 2$, the theorem holds by induction. Thus $g'' = h$ and $N'' + 1 < N$. Therefore again by induction the theorem is true. This completes the proof.

Chapter 8

Fusion Rules, Fusion Rings and Verlinde Formula

In this section we talk about fusion rules and study in details the most relevant example of fusion rules coming from the dimension of conformal blocks on a N-pointed stable curve. Using the fusion rules we define the Fusion ring $\mathcal{R}_\ell(\mathfrak{g})$ and subsequently move on to give a proof of Verlinde formula. We mostly follow the descriptions of [Beau].

8.1 Fusion rules

Let us recall the Grothendieck ring $\mathcal{R}(\mathfrak{g})$ of finite dimensional $\mathfrak{g}$-modules for a simple Lie algebra $\mathfrak{g}$. The elements of the ring are isomorphism classes of finite dimensional $\mathfrak{g}$-modules. We denote the isomorphism class of a $\mathfrak{g}$-module V by $[V]$. Let $[V_\lambda]$ denote the isomorphism class of a irreducible finite dimensional $\mathfrak{g}$ with highest weight λ. It turns out that $\mathcal{R}(\mathfrak{g})$ is a $\mathbb{Z}$-module generated with basis as $[V_\lambda]$. The multiplication is defined by,

$$[V_\lambda].[V_\mu] = [V_\lambda \otimes V_\mu].$$

Thus $\mathcal{R}(\mathfrak{g})$ is a commutative ring.

For each $\mathfrak{g}$-module V, the dual module V^* is defined to be $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. The action of $\mathfrak{g}$ on V^* is given by,

$$X.f(v) = -f(X.v);$$

where $X \in \mathfrak{g}$ and $v \in V$. In particular if V is simple, then V^* is also a simple $\mathfrak{g}$-module. For $\lambda \in P_+$, the dual $\mathfrak{g}$-module is V_{λ^*} of highest weight λ^*. $\lambda^* = -w(\lambda)$, where w is the longest element in the Weyl group. The map

$$\lambda \longrightarrow \lambda^*,$$

induces a $\mathbb{Z}$ linear involution on P. This also induces an involution of the root system and it preserves the long root θ. Thus P_ℓ is preserved under the involution.

The following lemma is trivial.

Lemma 32. *There exists a unique involution automorphism of $\mathcal{R}(\mathfrak{g})$ given by $\lambda \to$ lambda* such that for every element $[V]$ in $\mathcal{R}(\mathfrak{g})$ we get*

$$[V]^* = [V^*]$$

We prove a useful fact about conformal blocks which we will use in later while discussing fusion rules.

Proposition 13. *Start with a N-pointed stable curve C with the following data:*

$$\mathfrak{X} = (C; Q_1, Q_2, \cdots, Q_N; \eta_1, \eta_2 \cdots, \eta_N).$$

Let $\vec{\lambda^*} = (\lambda_1^*, \lambda_2^*, \cdots, \lambda_N^*)$; where $\lambda \to \lambda^*$ is the involution. Then there is a natural isomorphism between

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) \simeq \mathcal{V}_{\vec{\lambda^*}}(\mathfrak{X}).$$

Proof. Using the above lemma, we see that there is a automorphism σ of $\mathfrak{g}$ such that for all finite dimensional representations ρ of $\mathfrak{g}$, we have $\rho \circ \sigma$ is isomorphic to ρ^*. The automorphism of $\mathfrak{g}$ extends to a automorphism $\widehat{\sigma}$ of $\widehat{\mathfrak{g}}$, which preserves the decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus \mathbb{C}c \oplus \widehat{\mathfrak{g}}_+,$$

where we define $\widehat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \xi\mathbb{C}[[\xi]]$ and $\widetilde{\mathfrak{g}}_- = \mathfrak{g} \otimes \xi^{-1}\mathbb{C}[\xi^{-1}]$ as subspaces of $\widehat{\mathfrak{g}}$. For $\lambda \in P_\ell$ consider the representation

$$\widehat{\rho_\lambda} : \widehat{\mathfrak{g}} \longrightarrow \mathrm{End}(\mathcal{H}_\lambda).$$

Then the representation $\widehat{\rho_\lambda} \circ \widehat{\sigma}$ is also simple. The subspace of $\mathcal{H}_\lambda$ annihilated by $\widehat{\mathfrak{g}}_+$ is V_λ. On V_λ the action of $\mathfrak{g}$ is by $\rho_\lambda \circ \sigma$, therefore $\widehat{\rho_\lambda} \circ \widehat{\sigma}$ is isomorphic to $\widehat{\rho}_{\lambda^*}$. Thus there is a $\mathbb{C}$-linear isomorphism

$$T_\lambda : \mathcal{H}_\lambda \longrightarrow \mathcal{H}_{\lambda^*},$$

such that for $X \in \widehat{\mathfrak{g}}$ and $v \in \mathcal{H}_\lambda$

$$T_\lambda(X.v) = \widehat{\sigma}(X).T_\lambda(v).$$

Now we define the map

$$\begin{aligned} T_{\vec{\lambda}} &: \mathcal{H}_{\vec{\lambda}} \to \mathcal{H}_{\vec{\lambda^*}} \\ T_{\vec{\lambda}} &:= t_{\lambda_1} \otimes t_{\lambda_2} \otimes \cdots \otimes t_{\lambda_N}. \end{aligned}$$

It follows that for $X \in \mathfrak{g}$ and $f \in H^0(C, \mathcal{O}_C(* \sum_{j=1}^N Q_j))$;

$$T_{\vec{\lambda}}(X \otimes f)(v) = (\sigma(X) \otimes f) T_{\vec{\lambda}}(v).$$

Thus $T_{\vec{\lambda}}$ induces a $\mathbb{C}$-linear isomorphism of $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$ onto $\mathcal{V}_{\vec{\lambda^*}}(\mathfrak{X})$. □

Let us denote by $N_g(\vec{\lambda})$, the dimension of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$; where the underlying curve has genus g. As we have already proved that the dimension depends only on the genus of the curve and $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_N)$. The above proposition tell us,

$$N_g(\vec{\lambda}) = N_g(\vec{\lambda}^*).$$

We start with a genus g curve and we deform it to a nodal curve of genus g and consider its normalization which has genus $g-1$. Since the dimension of the conformal block does not change under deformations, by using the factorization theorem, we have

$$N_g(\vec{\lambda}) = \sum_{\nu \in P_\ell} N_{g-1}(\vec{\lambda}, \nu, \nu^*). \tag{8.1}$$

Let $\vec{\mu} = (\mu_1, \mu_2, \cdots, \mu_M)$ be another set of points. Then for the $N+M$ pointed curve C of genus g we consider the deformation of the curve C into a nodal curve whose components are N and M pointed curve of genus h and k. Moreover $h+k=g$, where h and k are positive integers less than g. By factorization we have,

$$N_g(\vec{\lambda}, \vec{\mu}) = \sum_{\nu \in P_\ell} N_h(\vec{\lambda}, \nu) N_k(\vec{\mu}, \nu^*). \tag{8.2}$$

For the genus $g = 0$ case, we have shown that

$$N_0(\lambda) = 0;$$

for $\lambda \neq 0$ and $N_0(0) = 1$. Further

$$N_0(\lambda, \mu) = 0$$

unless $\mu = \lambda^*$ and $N_0(\lambda, \lambda^*) = 1$.

Let I be a finite set with an involution $\lambda \to \lambda^*$. We denote by $\mathbb{N}^I$, the free commutative monoid generated by the set I. It is the set of sums of the form $\sum_{\alpha \in I} n_\alpha \alpha$ with $n_\alpha \in \mathbb{N}$. We extend the involution of I by linearity to an involution on $\mathbb{N}^I$.

Definition: A *fusion rule* on I is a map

$$N : \mathbb{N}^I \longrightarrow \mathbb{Z},$$

such that

(1) $N(0) = 1$ and $N(\alpha) > 0$ for some $\alpha \in I$,

(2) $N(x^*) = N(x)$ for every $x \in \mathbb{N}^I$,

(3) For all x and y in $\mathbb{N}^I$, one has

$$N(x+y) = \sum_{\lambda \in I} N(x+\lambda) N(y+\lambda^*).$$

We define the *Kernel* of the fusion rule to be the set of all elements $\alpha \in I$ such that

$$N(\alpha + x) = 0,$$

for all $x \in \mathbb{N}^I$. The fusion rule is said to be nondegenerate, if the kernel N is trivial. Let N is a fusion rule, we can consider the restriction of the fusion rule on $\mathbb{N}^{I-K}$. Then the restricted fusion rule N_0 is nondegenerate. In this section we will restrict ourselves to nondegenerate fusion rules. Next we consider the most relevant example of the fusion rule.

For a simple complex Lie algebra $\mathfrak{g}$ and a level ℓ. For $I = P_\ell$ and for weights $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ consider the fusion rule defined by,

$$N(\sum \lambda_i) = \dim_{\mathbb{C}}(\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}));$$

where $\mathfrak{X}$ is the N-pointed projective line $\mathbb{P}^1$ and λ_i's are the weights associated to the point Q_i on $\mathbb{P}^1$. It is clear from the previous discussion that N is a nondegenerate fusion rule on P_ℓ.

For a commutative ring R with a involutive ring homomorphism and a $\mathbb{Z}$-linear form t on R such that the bilinear form

$$(x, y) = t(xy^*)$$

is symmetric. I be a orthonormal basis containing 1, we define a map

$$N : \mathbb{N}^I \longrightarrow Z$$

by,

$$N(\sum n_\beta \beta) = t(\prod \beta^{n_\beta}).$$

Then N is a nondegenerate fusion rule on I and in particular we have $t(x^*) = t(x)$ and for $x, y \in R$ we have,

$$t(xy) = \sum_{\lambda \in I} t(x\lambda)t(y\lambda^*).$$

We have a converse in the following proposition.

Proposition 14. *Let $N : \mathbb{N}^I \to \mathbb{Z}$ be a nondegenerate fusion rule on I. Then there exists a $\mathbb{Z}$-bilinear map*

$$\mathbb{Z}^I \times \mathbb{Z}^I \longrightarrow \mathbb{Z}^I$$

which turns $\mathbb{Z}^I$ into a commutative ring and a linear form t uniquely determined such that

$$N(\sum n_\beta \beta) = t(\prod \beta^{n_\beta})$$

for all elements $\sum n_\beta \beta \in \mathbb{N}^I$. Also, we have

$$t(\alpha, \beta^*) = \delta_{\alpha,\beta};$$

for $\alpha, \beta \in I$.

Proof. Putting $x = y = 0$ in the first rule, we get

$$\sum_{\lambda \in I} N(\lambda)^2 = 1.$$

Since $N(\lambda)$ takes values in the integers; there exists an element $\epsilon \in I$ such that $N(\epsilon) = 1$ and $\epsilon = \epsilon^*$ we also have $N(\lambda) = 0$ for $\lambda \neq \epsilon$. Using the second fusion rule and putting $y = 0$, we see that

$$\begin{aligned} N(x) &= \sum_{\lambda \in P_\ell} N(x+\lambda)N(\lambda^*) \\ &= N(x+\epsilon)N(\epsilon) \\ &= N(x+\epsilon) \quad \text{(Propagation of Vacua).} \end{aligned}$$

With $x = \alpha$ and $y = \alpha^*$ and using rule 3, we get

$$N(\alpha + \alpha^*) = \sum_{\lambda \in I} N(\alpha + \lambda)^2 \geq N(\alpha + \alpha^*)^2.$$

Now if $N(\alpha + \lambda) = 0$ for all $\lambda \in I$, then we see from the third fusion rule that

$$N(\alpha + x) = 0$$

for all $x \in \mathbb{N}^I$. This is a contradiction since N is nondegenerate. Thus the only possibility is that $N(\alpha + \alpha^*) = 1$ and $N(\alpha + \lambda) = 0$ if, $\lambda \neq \alpha^*$.

Now we are ready to define a multiplication on the ring $\mathbb{Z}^I$ as follows

$$\alpha.\beta = \sum_{\lambda \in I} N(\alpha + \beta + \lambda^*)\lambda$$

and extend the definition by bilinearlity. The law is clearly commutative and we next show that it is also associative. For $\alpha, \beta, \gamma \in I$ we have,

$$\begin{aligned} (\alpha.\beta).\gamma &= \big(\sum_{\lambda \in I} N(\alpha + \beta + \lambda^*)\lambda\big).\gamma \\ &= \sum_{\lambda \in I}\sum_{\mu \in I} N(\alpha + \beta + \lambda^*)N(\lambda + \gamma + \mu^*)\mu \\ &= \sum_{\mu \in I}\sum_{\lambda \in I} N(\beta + \gamma + \lambda^*)N(\alpha + \lambda + \mu^*)\mu \\ &= \big(\sum_{\lambda \in I} N(\beta + \gamma + \lambda^*)\big)\alpha.\lambda \\ &= \alpha.\big(\sum_{\lambda \in I} N(\beta + \gamma + \lambda^*)\lambda\big) \\ &= \alpha.(\beta.\gamma). \end{aligned}$$

By induction on N, for $\alpha_i \in I$, we get the following:

$$\alpha_1.\alpha_2 \cdots \alpha_N = \sum_{\lambda \in I} N(\alpha_1 + \alpha_2 + \cdots + \alpha_N + \lambda^*)\lambda.$$

Again,

$$\epsilon.\alpha = \sum_{\lambda \in I} N(\epsilon + \alpha + \lambda^*)\lambda = \alpha.$$

$N(x^*) = N(x)$ implies, $x \to x^*$ is a ring homomorphism.

$$t : \mathbb{Z}^I \longrightarrow \mathbb{Z},$$

is defined as

$$\sum n_\alpha \alpha \to n_\epsilon.$$

From the previous discussion we see,

$$t(\alpha_1.\alpha_2.\cdots \alpha_N) = N(\alpha_1 + \alpha_2 + \cdots + \alpha_N).$$

Since $N(\alpha + \lambda) = 0$, unless $\alpha = \beta^*$ and $N(\alpha + \alpha^*) = 1$. Thus, we get

$$t(\alpha.\beta^*) = \delta_{\alpha,\beta}.$$

□

Definition The ring $\mathbb{Z}^I$ with the multiplication defined in the previous proposition is called the fusion ring associated to N. We denote it by $\mathcal{F}_N$ or simply $\mathcal{F}$. Now, the bilinear form $(x, y) = t(xy)$ defines an isomorphism of $\mathcal{F}$ with the $\mathcal{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{F}, \mathbb{Z})$. The element Tr in $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{F}, \mathbb{Z})$ is defined by the trace of the endomorphism induced by, multiplication L_x by x. We claim that the element corresponding to Tr in $\mathcal{F}$ is the Casimir element c defined by,

$$c = \sum_{\lambda \in I} \lambda\lambda^*.$$

We know that

$$x.y = \sum_{\lambda \in I} N(x + y + \lambda^*)\lambda.$$

Thus,

$$\begin{aligned}
\mathrm{Tr}(x) &= \sum_{\lambda \in I} N(x + \lambda + \lambda^*) \\
&= \sum_{\lambda \in I} t(\lambda\lambda^* x) \\
&= t(cx).
\end{aligned}$$

Proposition 15. *Given a sequence of maps*

$$N_g : \mathbb{N}^I \longrightarrow \mathbb{Z},$$

such that, $N_0 = N$ *and*

$$N_g(x) = \sum_{\lambda \in I} N_{g-1}(x + \lambda + \lambda^*),$$

for all $x \in \mathbb{N}^I$ *and* $g \geq 1$*. Then, for* $\alpha_1, \alpha_2, \cdots, \alpha_m \in I$,

$$N_g(\alpha_1 + \alpha_2 + \cdots + \alpha_m) = t(\alpha_1\alpha_2 \cdots \alpha_m c^g) = Tr(\alpha_1\alpha_2 \cdots \alpha_m c^{g-1}).$$

Proof.

$$\begin{aligned}
N_g(\alpha_1+\cdots+\alpha_m) &= \sum_{\lambda_1\in I} N_{g-1}(\alpha_1+\cdots+\alpha_m+\lambda_1+\lambda_1^*)\\
&= \sum_{\lambda_1\in I}\sum_{\lambda_2\in I} N_{g-2}(\alpha_1+\cdots+\alpha_m+\lambda_1+\lambda_1^*+\lambda_2^*)\\
&= \sum_{\lambda_1,\lambda_2,\cdots,\lambda_g\in I} N_0(\alpha_1+\cdots+\alpha_m+\lambda_1+\cdots+\lambda_g+\lambda_1^*+\cdots+\lambda_g^*)\\
&= \sum_{\lambda_1,\lambda_2,\cdots,\lambda_g\in I} t(\alpha_1\alpha_2\cdots\alpha_m\lambda_1\lambda_2\cdots\lambda_g\lambda_1^*\lambda_2^*\cdots\lambda_g^*)\\
&= t(\alpha_1\cdots\alpha_m c^g)\\
&= \mathrm{Tr}(\alpha_1\cdots\alpha_m c^{g-1}).
\end{aligned}$$

□

Remark: The above defined linear form t gives rise to a canonical bilinear form:

$$\mathbb{N}^I \times \mathbb{N}^I \longrightarrow \mathbb{Z}$$

define by $(x,y)=t(xy^*)$, such that I is an orthonormal basis with respect to the bilinear form. Since $*$ is a ring homomorphism for $x,y,z\in\mathcal{F}$ one gets,

$$(xy,z)=(x,y^*z).$$

This imposes further restriction on the ring $\mathcal{F}$.

We state without proof a proposition and corollary of the proposition. For a proof see [Beau].

Proposition 16. *The $\mathbb{Q}$ algebra $\mathcal{F}_{\mathbb{Q}}=\mathcal{F}\otimes\mathbb{Q}$ is isomorphic to a product $\prod K_i$ of finite extensions of $\mathbb{Q}$, preserved by the involution, which are of the following two types:*

(a) a totally real extension of $\mathbb{Q}$ with the trivial involution,

(b) a totally imaginary extension of $\mathbb{Q}$ which is a quadratic extension of a totally real extension of $\mathbb{Q}$, the involution being the nontrivial automorphism of that quadratic extension.

A character χ of the ring $\mathcal{F}$ to $\mathbb{C}$ is defined to be the algebra homomorphism of $\mathcal{F}$ to $\mathbb{C}$. We denote by S the set of characters of the fusion ring $\mathcal{F}$ to $\mathbb{C}$. By the previous theorem, we get:

Corollary 8. *(a) The map*

$$\mathcal{F}_{\mathbb{C}} \longrightarrow \mathbb{C}^S$$

given by,

$$x \longrightarrow (\chi(x))_{\chi\in S}$$

is an isomorphism of $\mathbb{C}$-algebras and

(b) $\chi(x^*)=\overline{\chi(x)}$ for $\chi\in S$ and $x\in\mathcal{F}$.

Now by the above isomorphism, studying the fusion ring is reduced to studying the characters on the fusion ring. With the same notation as that of the proposition on N_g we have,

Proposition 17.

$$N_g(\alpha_1 + \alpha_2 + \cdots + \alpha_N) = \sum_{\chi \in S} \chi(\alpha_1)\chi(\alpha_2)\cdots\chi(\alpha_N)\chi(c)^{g-1};$$

where $\chi(c) = \sum_{\lambda \in I} |\chi(\lambda)|^2$.

Proof. For $x \in \mathcal{F}$ the corresponding element in $\mathbb{C}^S$ is $(\chi(x)_{\chi \in S})$. Thus multiplication by x is same as the multiplication by the diagonal matrix whose entries are $(\chi(x)_{\chi \in S})$. Thus

$$\mathrm{Tr(x)} = \sum_{\chi \in S} \chi(x).$$

Hence,

$$\begin{aligned} N_g(\alpha_1 + \cdots + \alpha_m) &= \mathrm{Tr}(\alpha_1 \cdots \alpha_m c^{g-1}) \\ &= \sum_{\chi \in S} \chi(\alpha_1)\cdots\chi(\alpha_m)\chi(c)^{g-1}. \end{aligned}$$

□

8.2 Fusion rings

Earlier we talked about the Grothendieck ring of finite dimensional representations of $\mathfrak{g}$. It has a multiplication structure induced by the tensor product of $\mathfrak{g}$-modules. Now, we consider the analogue of $\mathcal{R}(\mathfrak{g})$ for level ℓ representations of $\widehat{\mathfrak{g}}$. But the multiplication structure can not be given by simply taking the tensor product as the tensor product of two level ℓ representations has level 2ℓ. Instead we use the fusion rule to define a multiplicative structure.

To a simple Lie algebra $\mathfrak{g}$ we have associated a fusion rule

$$N(\sum \lambda_i) = \dim_{\mathbb{C}} \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X});$$

where $\mathfrak{X}$ is the data associated to the N-pointed projective line $\mathbb{P}^1$.

Let $\mathcal{R}_\ell(\mathfrak{g})$ be the $\mathbb{Z}$-module generated with $[V_\lambda]$ as basis where $\lambda \in P_\ell$. We define the multiplication

$$[V_\lambda].[V_\beta] = \sum_{\mu \in P_\ell} N(\lambda + \beta + \mu^*)[V_\mu].$$

Under this multiplication, $\mathcal{R}_\ell(\mathfrak{g})$ gets the structure of a commutative ring. It is a subgroup of $\mathcal{R}(\mathfrak{g})$ but not a subring since the multiplication in $\mathcal{R}_\ell(\mathfrak{g})$ is completely different form the multiplication in $\mathcal{R}(\mathfrak{g})$.

We recall some basic facts about Weyl chambers and dominant weights. Let P denote the weight lattice for the Lie algebra $\mathfrak{g}$. For each root α the equation $\lambda(H_\alpha) = 0$ defines a hyperplane, known as the walls in the vector space $P \otimes \mathbb{R}$. The connected components of the compliments of the wall are known as chambers. The chambers are fundamental domains for the Weyl group action on $P \otimes \mathbb{R}$. To a basis $(\alpha_1, \alpha_2, \cdots, \alpha_r)$ of a root system Δ, we assign the chamber C, known as the Weyl chamber. It is defined by the conditions $\lambda(H_{\alpha_i}) \geq 0$. The weight which lie in the interior of the Weyl chamber are known as dominant regular weights. Since C is a fundamental domain for the action of the Weyl groups, every element of P can be written as a $w\lambda_+$; where $w \in W$ and $\lambda_+ \in P_+$. The stabilizer of an element belonging to the interior of the Weyl chamber is trivial. Let ρ be the half sum of positive roots. Since

$$\rho(H_{\alpha_i}) = 1;$$

$\lambda_+ + \rho$ belongs to the interior of the Weyl chamber, for any $\lambda_+ \in P_+$. Moreover the weights which belong to the interior of the Weyl chamber are of the form $\lambda_+ + \rho$.

For studying the ring $\mathcal{R}_\ell(\mathfrak{g})$, we consider a analogue of the Weyl group known as the affine Weyl group W_ℓ. The affine Weyl group W_ℓ is generated by W and translations of the form

$$x \longrightarrow x + (\ell + g^*)\theta;$$

where θ is a long root and $g^* = 2\rho(H_\theta) + 1$ is the dual Coxeter number. Since each long root is conjugate to θ under the action of the Weyl group W, we see that

$$W_\ell = W \ltimes (\ell + g^*)Q_{lg;}$$

where Q_{lg} is the sublattice of P spanned by the long roots. Here the affine walls are defined by the relations

$$\lambda(H_\alpha) = (\ell + g^*)n.$$

for each root α and $n \in \mathbb{Z}$. The connected components of the compliment of the affine walls are called affine chambers. As in the case of the Weyl group the affine chambers are fundamental domain for the action of W_ℓ on $P \otimes \mathbb{R}$. The alcove A contained in C and containing 0 is defined by the equation

$$\lambda(H_{\alpha_i}) \geq 0,$$

for each element α_i in the basis and $\lambda(H_\theta) \leq (\ell + g^*)$. As before the weights belonging to the interior of the alcove A are of the form $\lambda_+ + \rho$ where $\lambda_+ \in P_\ell$.

Let $\mathbb{Z}[P]$ denote the group ring of P generated by elements of the form e^λ for $\lambda \in P$. The multiplication of the basis is given by,

$$e^\lambda . e^\beta = e^{\lambda + \beta}.$$

We extend the multiplication by $\mathbb{Z}$-linearity. Let $\mathbb{Z}[P]_W$ denote the quotient of $\mathbb{Z}[P]$ by the lattice generated by elements of the form:

(a) $e^\lambda - sgn(w)e^{w\lambda}$, where $\lambda \in P$, $w \in W$ and $sgn(w)$ denote the sign of the element w,

(b) e^λ, where λ belongs to the wall.

Observe that for λ in a wall we have,

$$2e^\lambda = e^\lambda - sgn(s_\alpha)e^{s_\alpha(\lambda)}.$$

Similarly we define $\mathbb{Z}[P]_{W_\ell}$, where we replace the W by W_ℓ in the above definition. We have the following lemma:

Lemma 33. *The linear maps,*

$$\phi : \mathcal{R}(\mathfrak{g}) \longrightarrow \mathbb{Z}[P]_W \quad \textit{and} \quad \phi_\ell : \mathcal{R}_\ell(\mathfrak{g}) \longrightarrow \mathbb{Z}[P]_{W_\ell}$$

defined by, $[V_\lambda] \to e^{\lambda+\rho}$ *is bijective.*

Proof. We prove this by constructing a map in the opposite direction. We define a linear map,

$$\psi : \mathbb{Z}[P] \longrightarrow \mathcal{R}(\mathfrak{g})$$

in the following way. If $\lambda \in P$ does not lie on the wall, then it is of the form $w(\lambda_+ + \rho)$ where $\lambda_+ \in P_+$, $w \in W$ are uniquely determined. We define,

$$\psi(e^\lambda) = \begin{cases} sgn(w)[V_{\lambda_+}] & \text{if } \lambda \text{ does not belong to the wall.} \\ 0 & \text{otherwise.} \end{cases}$$

The map ψ factors through the $\widehat{\psi} : \mathbb{Z}[P]_W \to \mathcal{R}(\mathfrak{g})$. It is easy to show that ψ is inverse of the map ϕ. □

By the above lemma we see there is a map $\pi : \mathcal{R}(\mathfrak{g}) \to \mathcal{R}_\ell(\mathfrak{g})$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{R}(\mathfrak{g}) & \xrightarrow{\pi} & \mathcal{R}_\ell(\mathfrak{g}) \\ \downarrow \wr\, \phi & & \downarrow \wr\, \phi_\ell \\ \mathbb{Z}[P]_W & \xrightarrow{p} & \mathbb{Z}[P]_{W_\ell} \end{array}$$

where the map p is the projection map. From the proof of the previous lemma we also see that,

$$\begin{aligned} \pi([V_\lambda]) &= 0 \quad \text{if } \lambda + \rho \text{ belongs to the affine wall,} \\ \pi([V_\lambda]) &= [V_\lambda] \quad \text{for } \lambda \in P_\ell. \end{aligned}$$

We are interested in the characters of the fusion ring $\mathcal{R}_\ell(\mathfrak{g})$. For convenience, we replace $\mathfrak{g}$ by the simply connected Lie group G whose Lie algebra is $\mathfrak{g}$. We also replace $\mathfrak{h}$ by T, the maximal torus in G. Any element $\lambda \in P$ defines a character e^λ on T as follows. Since any element of T can be written in the form $exp(H)$, for some $H \in \mathfrak{h}$,

$$e^\lambda(exp(H)) = e^{\lambda(H)}.$$

This defines an isomorphism of the character group of T with P.

For each t of T defines a character $\mathrm{Tr}_*(t)$ on $\mathcal{R}(\mathfrak{g})$. It takes a $\mathfrak{g}$-module V and assigns the number $\mathrm{Tr}_V(t)$. The *Weyl character formula* gives us an explicit way of computing Tr. We introduce the anti-symmetrization operator as,

$$J(e^{\lambda}) = \sum_{w \in W} sgn(w) e^{w\lambda}.$$

One has,

$$J(e^{\rho}) = e^{\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})$$

We define an element t of T to be a *regular* element, if

$$e^{\alpha}(t) \neq 1,$$

for each root α. Equivalently the action of the Weyl group on t has trivial stabilizer. The *Weyl character formula* tell us that, for a regular element t;

$$\mathrm{Tr}_{V_\lambda}(t) = \frac{J(e^{\lambda+\rho})(t)}{J(e^{\rho})(t)}.$$

Now we define the analogue of regular elements for the case of the affine Weyl group. Let T_ℓ denotes the set of all elements $t \in T$, such that $e^{\alpha}(t) = 1$ for each $\alpha \in (\ell + g^*)Q_{lg}$. T_ℓ^{reg} is the regular elements in T_ℓ. The elements of T_ℓ^{reg} play a key role in determining the characters of the ring $\mathcal{R}_\ell(\mathfrak{g})$.

Lemma 34. *With the above set up we have,*

(1) For $t \in T_\ell^{reg}$, the character $Tr_*(t)$ factors thorough $\pi : \mathcal{R}(\mathfrak{g}) \to \mathcal{R}_\ell(\mathfrak{g})$,

(2) We identify $P \otimes \mathbb{C}$ with $\mathfrak{h}$ using the normalized Cartan Killing form. Then the map,

$$\lambda \longrightarrow \exp 2\pi i \frac{\lambda}{\ell + g^*}$$

induces an isomorphism of $P/(\ell + g^*)Q_{lg}$ onto T_ℓ,

(3) The map $\lambda \to \exp 2\pi i \frac{\lambda+\rho}{\ell+g^*}$, induces a bijection of P_ℓ onto T_ℓ^{reg}/W.

Proof. (1) Let $t \in T_\ell^{reg}$. Let us define the map j_t by,

$$j_t(e^{\mu}) = \frac{J(e^{\mu})(t)}{J(e^{\rho})(t)}.$$

The Weyl character formula gives us the following diagram:

$$\begin{array}{ccc} \mathcal{R}(\mathfrak{g}) & & \\ \wr\downarrow \phi & \searrow^{\mathrm{Tr}_*(t)} & \\ \mathbb{Z}[P]_W & \xrightarrow{j_t} & \mathbb{C}. \end{array}$$

We have to show, j_t kills all elements of the kernel p. The kernel of the map π corresponds to the kernel of the map p. The kernel of p is generated by elements of the form $e^{\mu} -$

$sgn(w_\ell)e^{w\mu+\alpha}$, where $\alpha \in (\ell+g^*)Q_{lg}$ and also by element of the form e^μ for some μ in the affine wall. Since $w_\ell = w \ltimes \alpha$, we get, $sgn(w_\ell) = sgn(w)$. Since $t \in T_\ell^{reg}$, we know $e^\alpha(t) = 1$. Thus,

$$\begin{aligned}
J(sgn(w)e^{w\mu+\alpha})(t) &= \sum_{w'\in W} sgn(w')sgn(w)e^{w'(w\mu+\alpha)}(t) \\
&= \sum_{w'\in W} sgn(w'w)e^{w'w(\mu)}(t) \\
&= \sum_{w\in W} sgn(w)e^{w(\mu)}(t) \\
&= J(e^\mu).
\end{aligned}$$

So, j_t kills elements of first kind. By a previous remark, $2e^\mu$ is an element of the first kind. Also $2j_t(e^\mu) = 0$. We get, $j_t(e^\mu) = 0$. This completes the proof of (1).

For (2) we consider the exact sequence:

$$0 \longrightarrow 2\pi i Q^\vee \longrightarrow \mathfrak{h} \longrightarrow T \longrightarrow 0;$$

where $Q^\vee$ is the coroot lattice of the root system of $\mathfrak{g}$. Let us denote by $P_{lg}^\vee \subset Q^\vee \otimes \mathbb{Q}$, to be the set of elements H, such that for all $\alpha \in Q_{lg}$:

$$\alpha(H) \in \mathbb{Z}.$$

The map,

$$H \longrightarrow \exp\left(\frac{2\pi i}{\ell+g^*}H\right),$$

induces an isomorphism of $P_{lg}^\vee/(\ell+g^*)Q^\vee$ onto T_ℓ. After identifying $Q^\vee \otimes \mathbb{Q}$ with $P \otimes \mathbb{Q}$ using the normalized Cartan Killing form, we see $P_{lg}^\vee$ is identified with the dual lattice of Q_{lg}. In other words, the set of all elements λ in $P \otimes \mathbb{Q}$, such that $\langle \lambda, \beta \rangle \in \mathbb{Z}$ for each long root β. This is equivalent to $\lambda(H_\beta) \in \mathbb{Z}$. But the elements H_β span the coroot lattice $Q^\vee$. Thus we see that the dual of Q_{lg} is P. Similarly we see, $Q^\vee$ identified with the dual lattice Q_{lg} of P.

(3) The isomorphism of $P/(\ell+g^*)Q_{lg} \simeq T_\ell$ is equivariant under the action of the Weyl group W. The Weyl group orbits of $P/(\ell+g^*)Q_{lg}$ are in one-one correspondence with the affine Weyl group orbits in P. These are the elements of P which lie in the affine alcove. The orbits where W acts freely corresponds to the weights which lie in the interior of the alcove. This completes the proof. □

We still have the same notation $\mathrm{Tr}_*(t)$, for $t \in T_\ell^{reg}$, obtained by passing to the quotient of the map π. If $\lambda \in P_\ell$ and it does not belong to any affine wall, $\pi([V_\lambda]) = [V_\lambda]$. The map $\mathrm{Tr}_*(t)$ only depends upon the class of $t \in T_\ell^{reg}/W$. We state without proof a result from [Fal].

Lemma 35. *The following conditions are equivalent:*

(1) The map $\pi : \mathcal{R}(\mathfrak{g}) \to \mathcal{R}_\ell(\mathfrak{g})$ is a ring homomorphism.

(2) $\pi([V_\lambda \otimes V_\omega]) = [V_\lambda].[V_\omega]$, for each $\lambda \in P_\ell$ and each fundamental weight $\omega \in P_\ell$.

(3) The linear forms $Tr_*(t)$ for $t \in T_\ell^{reg}/W$ are characters of the fusion ring $\mathcal{R}_\ell(\mathfrak{g})$.

When these conditions hold, the characters of $\mathcal{R}_\ell(\mathfrak{g})$ are of the form $Tr_*(t)$, for $t \in T_\ell^{reg}/W$.

We state another lemma without proof from [Fal]:

Lemma 36. *The above conditions holds when $\mathfrak{g}$ is of type A_r, B_r, C_r, D_r and G_2.*

The proof for other exceptional Lie algebras can be found in [Tel].

8.3 Verlinde formula

To compute the dimension of the conformal blocks we only need to know how to compute the trace of the Casimir element $\sum_{\lambda \in P_\ell} |\chi(\lambda)|^2$. We have the following:

Proposition 18. *Let $t \in T_\ell^{reg}$. Then,*

$$\sum_{\lambda \in P_\ell} |\, Tr_{V_\lambda}(t)|^2 = \frac{|T_\ell|}{\Delta(t)};$$

where

$$\Delta(t) = |J(E^\rho)(t)|^2 = \prod_{\alpha \in \Delta^+} (e^\alpha(t) - 1).$$

Proof. For $\lambda \in P_\ell$ we denote

$$t_\lambda = \exp 2\pi i \frac{\lambda + \rho}{\ell + g^*}.$$

The t_λ's form a system of representative of T_ℓ^{reg}/W. The Cartan Killing form is invariant under the action of the Weyl group. So, we have

$$\begin{aligned} J(e^{\lambda+\rho})(t_\mu) &= \sum_{w \in W} sgn(w) \exp 2\pi i \frac{\langle w(\lambda+\rho), \mu+\rho\rangle}{\ell + g^*} \\ &= J(e^{\mu+\rho})(t_\lambda). \end{aligned}$$

By the Weyl character formula we have,

$$\begin{aligned} \sum_{\lambda \in P_\ell} |\mathrm{Tr}_{V_\lambda}(t_\mu)|^2 &= \sum_{\lambda \in P_\ell} \frac{1}{\Delta(t_\mu)} |J(e^{\lambda+\rho})(t_\mu)|^2 \\ &= \frac{1}{\Delta(t_\mu)} \sum_{\lambda \in P_\ell} |J(e^{\mu+\rho})(t_\lambda)|^2. \end{aligned}$$

Define $h(t) = J(e^{\mu+\rho}(t))$ be a function on T. Observe that $h(wt) = sgn(w)h(t)$, for $w \in W$ and $t \in T$. Also note h vanishes on non regular elements of T. Since t is nonregular, there is an reflection s_α such that $s_\alpha(t) = t$. Using the fact that h is anti-invariant we get,

$$h(t) = h(s_\alpha(t)) = -h(t).$$

But $|h|^2$ is W invariant. Therefore,

$$\begin{aligned} ||J(e^{\mu+\rho})(t)||^2 &= \langle J(e^{\mu+\rho}), J(e^{\mu+\rho})\rangle \\ &= \frac{1}{|T_\ell|}\sum_{t\in T_\ell}|J(e^{\mu+\rho})(t)|^2 \\ &= \frac{|W|}{|T_\ell|}\sum_{\lambda\in P_\ell}|J(e^{\mu+\rho})(t_\lambda)|^2, \end{aligned}$$

where the norm is the standard L^2 norm of functions on the finite group T_ℓ. Hence,

$$\sum_{\lambda\in P_\ell}|J(e^{\mu+\rho})(t_\lambda)|^2 = \frac{|T_\ell|}{|W|}||J(e^{\mu+\rho})||^2.$$

We wish to show that the restriction of the characters $e^{w(\mu+\rho)}$ are all distinct, where w is an arbitrary element in the Weyl group. By the orthogonality relations for the characters of a finite group we get,

$$||J(e^{\mu+\rho})|| = |W|.$$

Suppose the character's are not distinct; then there exists distinct elements w and w' in the Weyl group W such that

$$\langle w(\mu+\rho) - w'(\mu+\rho), \lambda\rangle \in (\ell+g^*)\mathbb{Z}, \quad \text{for all } \lambda\in P.$$

But we have seen that the dual lattice of P is Q_{lg}. This is equivalent to the following:

$$\mu+\rho - w^{-1}w'(\mu+\rho) \in (\ell+g^*)Q_{lg}.$$

This implies the existence of a nontrivial element in W_ℓ such that, $(\mu+\rho)$ is fixed by it. This is a contradiction. □

An immediate corollary is the following:

Corollary 9. *(* Verlinde formula *) Let $\mathfrak{g}$ be a complex simple Lie algebra, and a N-pointed stable curve C of genus g.*

$$\mathfrak{X} = (C; Q_1, Q_2, \cdots, Q_N; \eta_1, \eta_2 \cdots, \eta_N)$$

be the associated data.

$$\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_N)$$

are the representations corresponding to the N points on the curve. Then one gets,

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) &= |T_\ell|^{g-1}\sum_{t\in T_\ell^{reg}/W}\frac{Tr_{V_{\vec{\lambda}}}(t)}{\Delta(t)^{g-1}} \\ &= |T_\ell|^{g-1}\sum_{\mu\in P_\ell} Tr_{V_{\vec{\lambda}}}(\exp 2\pi i\frac{\mu+\rho}{\ell+g^*})\prod_{\alpha\in\Delta^+}\left|2\sin\pi\frac{\langle\alpha,\mu+\rho\rangle}{\ell+g^*}\right|^{2-2g}. \end{aligned}$$

Proof. By proposition-17,

$$\begin{aligned}
\dim_{\mathbb{C}} \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}) &= \sum_{\chi \in S} \chi(\lambda_1)\chi(\lambda_2)\cdots\chi(\lambda_N)\chi(c)^{g-1} \\
&= |T_\ell|^{g-1} \sum_{t \in T_\ell^{reg}/W} \frac{\mathrm{Tr}_{V_{\vec{\lambda}}}(t)}{\Delta(t)^{g-1}} \\
&= |T_\ell|^{g-1} \sum_{\mu \in P_\ell} \mathrm{Tr}_{V_{\vec{\lambda}}}(\exp 2\pi i \frac{\mu+\rho}{\ell+g^*}) \prod_{\alpha \in \Delta^+} \left|2 \sin \pi \frac{\langle \alpha, \mu+\rho \rangle}{\ell+g^*}\right|^{2-2g}.
\end{aligned}$$

□

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