Summary of recent work.

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Broad Classification:

I work at the interface of **algebraic geometry** and **representation theory**. I would like to classify my research work broadly into the following three topics:

- Semi-orthogonal decomposition of derived categoreis, moduli of bundles, their toric degenerations and mirror partners/potentials.
- Twisted conformal blocks, Verlinde formula, strange/rank-level dualities and tensor categories.
- Hitchin connection for parabolic bundles, their identification with Tsuchiya-Ueno-Yamada/Wess-Zumino-Witten on conformal blocks, and their Hodge theoretic aspects in genus zero (Knizhnik-Zamolodchikov. equations)

Class in $K_0(Var)$ of moduli of rank two bundles

Let $M_C^-(2)$ (resp. $M_C^+(2)$) be the moduli space of stable (resp. semistable) bundles of rank two with fixed determinant of odd (resp. even) degree on C.

Theorem: Belmans-Galkin-M:20

The following identity holds in $K_0(Var)$.

$$[M_C^-(2)] = \mathbb{L}^{g-1}[\mathrm{Sym}^{g-1} C] + \sum_{i=0}^{g-2} (\mathbb{L}^i + \mathbb{L}^{3g-3-2i})[\mathrm{Sym}^{\mathrm{i}} C] + T,$$

where $\mathbb{L} = [\mathbb{A}^1]$ and $(1+\mathbb{L})T = 0$.

Remark: We get a similar identity in $K_0(dgCat)$ which suggests the following:

Conjectural semi-orthogonal decomposition

Based on earlier works of Bondal-Orlov:95, Narasimhan:15, Fonarev-Kuznetsov:18:

Conjecture: Belmans-Galkin-M, Narasimhan

Let C be a smooth curve of genus g

$$\mathbf{D}^b(M_C^-(2)) = \langle \mathbf{D}^b(pt), \mathbf{D}^b(pt), \mathbf{D}^b(C), \mathbf{D}^b(C), \cdots \\ \cdots, \mathbf{D}^b(\mathrm{Sym}^{g-2} C), \mathbf{D}^b(\mathrm{Sym}^{g-2} C), \mathbf{D}^b(\mathrm{Sym}^{g-1} C) \rangle.$$

Theorem: Belmans-M:19

$$\mathbf{D}^b(M_C^-(r)) = \langle \mathbf{D}^b(pt), \mathbf{D}^b(pt), \mathbf{D}^b(C), \mathbf{D}^b(C), \mathcal{B} \rangle,$$

where $M_C^-(r)$ is the moduli space of rank r bundles with fixed determinant of degree one.

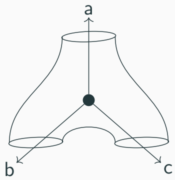
Remark: Lee-Moon has generalized BM:19 for any coprime degree.

Recent updates

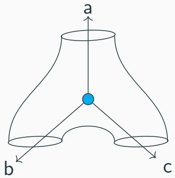
- Theorem: Lee-Narasimhan:21
  If C is not hyperelliptic, then
  $\mathbf{D}^b(M_C^-(2)) = \langle \mathbf{D}^b \operatorname{Sym}^2(C), \mathcal{C}' \rangle$.
- Theorem: Tevelev-Torres:21
  $\mathbf{D}^b(M_C^-(2)) = \langle \mathbf{D}^b(pt), \mathbf{D}^b(pt), \cdots, \mathbf{D}^b(\mathrm{Sym}^{\mathrm{g\text{-}1}}\mathrm{C}), \mathcal{A} \rangle$.
- Theorem: Xu-Yau:21
  $\mathbf{D}^b(M_C^-(2)) = \langle \{\Theta^\ell \otimes \mathbf{D}^b(\operatorname{Sym}^i(C))\}_{0 \le \ell < 2, i < g-\ell}, \mathcal{A}' \rangle$ with some generalizations for principal bundles.
- Theorem: Tevelev
  $\mathcal{A}$ as in Tevelev-Torres in trivial.

BGM-N conjecture via Laurent polynomials

$$W_{\bullet} = abc + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}$$



$$W_{\color{cyan}\bullet} = \frac{1}{abc} + \frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}$$



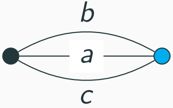
Graph Potential: Let (Γ, c) be a colored trivalent graph and $c : V(\Gamma) \to \{\pm 1\}$, define

$$W_{\Gamma,c} := \sum_{v \in V(\Gamma)} W_{v,c(v)}.$$

Question: Compute constant terms of powers of $W_{\Gamma,c}$?

$$\textbf{Period Series: } \sum_{m\geq 0} \frac{[W_{\Gamma,c}^m]_{const}}{m!} t^m = ??$$

Example: In $g = 2$:



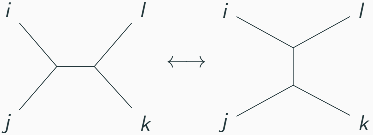
$$(b^2 a + \frac{2}{a} + \frac{a}{b^2}) + (\frac{1}{ac^2} + 2a + \frac{c^2}{a})$$

$$(abc + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}) + (\frac{1}{abc} + \frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c})$$

$$\textbf{Answer: } \sum_{m\geq 0} \frac{(2m!)^2}{(m!)^6} t^{2m}$$

Theorem (Belmans-Galkin-M:20)

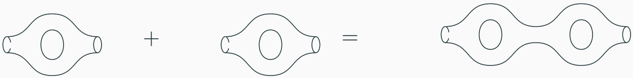
- The constant term $[(W_{\Gamma,c})^m]$ depends only on the genus g of Γ and total parity ϵ of the coloring c. In particular, a certain associativity constraint (WDVV equation) is satisfied.



- For Γ with no half edges (compact surfaces):

$$\sum_{m\geq 0} \frac{[(W_{\Gamma,c})^m]_{const}}{m!} t^m = \mathrm{Trace}(A^{g-1}S^{\epsilon+g}), \text{ where}$$

$S(x^n) := x^{-n}$ and $A = Bes(t(x+y)) \cdot Bes(t(x^{-1}+y^{-1}))$



B side: Graph potentials and $M^{\pm}_C(2)$

Theorem (Belmans-Galkin-M:20)

- *The moduli space $M^{-}_C(2)$ (resp $M^{+}_C(2)$) has a natural toric $X_{\Gamma,c}$ degeneration associated to a trivalent graph Γ whose Newton polynomial is the graph potential $W_{\Gamma,c}$.*

  Remark: *The degeneration (refining Manon:16) uses conformal blocks.*

- *If Γ has no separating edges, then $X_{\Gamma,c}$ has terminal singularities and hence*
- *(Kiem-Li:04) $M^{+}_C(2)$ has terminal singularities for a generic curve.*

A side: Mirror potentials and BGM-N Conjecture

Theorem (Belmans-Galkin-M:21-22)

- *The m-th descendent Gromov-Witten invariant of* $M_C^-(2)$ *is* $\frac{[(W_{\Gamma,c})^m]_{const}}{m!}$ *for any graph* (Γ, c) *of genus* g *with odd parity.*

  Remark: *Proposal of Eguchi-Hori-Xiong, for constructing mirror potential of Fano varieties. (Earlier: Abouzaid, Aroux, Coates-Corti-Galkin, FOOO, Givental, Konstevich, Katzarkov, Przylkowski, Nishinou-Nohara-Ueda, Orlov, Seidel).*

- *The set of critical values of* $W_{\Gamma,c}$

  $$\left\{-8(g-1), -8\sqrt{-1}(g-2), \ldots, 0, \ldots, 8\sqrt{-1}(g-2), 8(g-1)\right\}$$

  equals the eigen values (Muńoz:98) of quantum multiplication by $c_1(M_C^-(2))$*. The dimensions of the critical set with absolute critical value* $8(g-1-k)$ *is* k*.*

Conformal blocks and moduli of bundles for projective curves

Representation theory

- Let $\ell \in \mathbb{Z}_{>0}$ and consider the affine Lie algebra $\widehat{\mathfrak{g}}$.
- Let $(C, p_1, \ldots, p_n)$ be a stable curve and decorate each p_i with irreducibles $\lambda_i \in P_\ell(\mathfrak{g})$.

Algebraic geometry

- $\mathcal{M}^{par}_{G,\vec{\lambda}}(C)$ be the moduli of parabolic G-bundles with weights $(\lambda_1, \ldots, \lambda_n)$ on C.
- If $n = 0$, then $\mathcal{M}_G$ is a quotient of an affine Grassmanian by $G(H^0(\mathcal{O}_C(*p)))$.

Theorem/Definition (WZW, TUY, F, KNR, BL, P, LS, BF)

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{g}, \ell) := [\mathcal{H}_{\lambda_1} \otimes \ldots \mathcal{H}_{\lambda_n}]_{\mathfrak{g}\otimes H^0(\mathcal{O}_C(*\vec{p}))} \cong H^0(\mathcal{M}^{par}_{G,\vec{\lambda}}), \mathcal{L}_{\phi,\vec{\lambda}})^\vee.$$

Properties:Tsuchiya-Ueno-Yamada, Faltings, Teleman

- (Verlinde Formula) Conformal blocks give a vector bundle on $\overline{\mathcal{M}}_{g,n}$ and

$$\dim \mathcal{V}_{\vec{\lambda}}(C, \mathfrak{g}, \ell) = \sum_{\mu \in P_\ell(\mathfrak{g})} \frac{S_{\lambda_1,\mu} \cdots S_{\lambda_n,\mu}}{\left(S_{0,\mu}\right)^{n+2g-2}},$$

  where $S_{\mu,\lambda}$ is the S-matrix given explcitly in terms of transformations of characters.

- (TUY:89) Conformal blocks carry a flat projective connection known as the TUY/WZW/KZ connection.

- (TUY:89, Fakhruddin:12) In genus zero, it is a refinement of invariant of tensor products of representations

Strange/rank-level duality results

Rank-level duality predicts how for a pair $(\mathfrak{g}_1, \mathfrak{g}_2)$ of Lie algebras, the conformal blocks relate ? General approach to rank-level duality (M:16(i), M:16(ii)) for conformal pairs.

Theorem (M-Wentworth:19)

- *The monodromy representation on conformal blocks for* $\mathfrak{so}(2r+1)$ *with strictly spin representations at any odd level is reducible.*
- *The natural dualizing map is not an isomorphism.* $H^0(\mathcal{M}_{SO_{2r+1}}(C), \mathcal{P}^{\otimes 2s+1})^\vee \to H^0(\mathcal{M}_{SO_{2s+1}}(C), \mathcal{P}^{\otimes 2r+1})$, *where* $\mathcal{P}$ *is the Pfaffian bundle.*

(Earlier: Abe:08, Beauville-Narasimhan-Ramanan:94, Belkale:08-09, Boysal-Pauly:10, Marian-Oprea:09, Naculich-Warner, Nakanishi-Tsuchiya:94, Ostrik-Rowell:20).

Theorem: M-Zelaci:20

- Let η be a two torsion line bundle on C and let C_η be the étale double covering of C, then
$$H^0(Prym_\eta, (2r+1)\Xi_\eta)^\vee \cong H^0(\mathcal{M}_{SO(2r+1)}(C), \mathcal{D} \otimes \mathcal{L}_\eta),$$
(Pauly-Ramanan:01, Beauville:06).
- Further there is a flat identification,
$$H^0(SU^{\sigma,+}_{C_\eta}(2r+1), \mathcal{P}) \cong H^0(\mathcal{M}_{SO(2r+1)}(C), \mathcal{D} \otimes \mathcal{L}_\eta),$$
where $SU^{\sigma,+}_{C_\eta}(2r+1))$ is a higher rank (similarly for even case).

Remark: Degenerating dualities M:16 give relations among divisor classes in $\overline{\mathcal{M}}_{g,n}$. A survey of rank-level duality by me (M:21) has appeared as a chapter in a recent book Kumar:21.

Twist by a finite group Γ

Let $\pi : \widetilde{C} \to C$ be a Γ covering and $\Gamma \subset \mathrm{Aut}(\mathfrak{g})$.

- Preimage $\widetilde{p}_i$ in $\widetilde{C}$ of p_i,
- Γ_i be the stabilizer(cyclic) of $\widetilde{p}_i$ and Γ_i-twisted affine Lie algebras,
- Irreducibles $\lambda_i \in P_\ell(\mathfrak{g}, \Gamma_i)$ for each $\widetilde{p}_i$.

- (Balaji-Seshadri:15)Stack of (Γ, G)-bundles $\mathcal{B}un_{\Gamma,G}(\widetilde{C})$ of fixed local type on $\widetilde{C}$.
- (Pappas-Rapoport:08, Heinloth:10) Parahoric Bruhat-Tits group schemes $\mathcal{G}$ and their moduli $\mathcal{B}un_{\mathcal{G}}(C)$.

Twisted Covacua (Shen-Wang:01, Frenkel-Szczesny:04)

$$\mathcal{V}_{\vec{\lambda},\Gamma}(\widetilde{C}, C, \mathfrak{g}, \ell) := [\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}]_{(\mathfrak{g} \otimes (H^0(\widetilde{C} \backslash \Gamma.\widetilde{\mathbf{p}})))^\Gamma)}$$

Verlinde formula: (Conj for $\mathbb{Z}/2$ by Birke-Fuchs-Schweigert:02)

Theorem (Deshpande-M:19)

Let $\mathfrak{g}$ be a simple Lie algebra and let Γ preserve a Borel subalgebra (Hong-Kumar:18) of $\mathfrak{g}$. Then

$$\dim_{\mathbb{C}} \mathcal{V}_{\vec{\lambda},\Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}) = \sum_{\mu \in (P_\ell(\mathfrak{g}))^{\Gamma_0}} \frac{S^{m_1}_{\lambda_1,\mu} \dots S^{m_n}_{\lambda_n,\mu}}{(S_{0,\mu})^{n+2g-2}},$$

where Γ_0 is the image of a map from $\pi_1(C\backslash\mathbf{p}, *) \to \Gamma$ and m_i's are image of loops around the points p_i.

Moreover the γ-crossed S matrices are explicitly determined by transformation formulas (Kac-Wakimoto:88) of twisted Kac-Moody algebras.

Crossed categories via twisted conformal blocks

Theorem (Deshpande-M:19)

Γ-twisted conformal blocks define a Γ-crossed modular tensor category. The categorical crossed S matrices in the case are computed from characters of twisted Kac-Moody representations.

(This builds on works of E. Frenkel-Ben-Zvi:01, Kirillov jr, Damiolini:17, Hong-Kumar:18, Huang, M:16 and also works of Beilinson-Bernstein:81, Beilinson-Schechtman:88, I. Frenkel, Konstevich:87, Tsuchimoto:93 on Atiyah algebras and localization on $\mathcal{D}$-modules.)

Geometrization of TUY/WZW d'après Hitchin

Goal: Show that $\pi_*\mathcal{L}_{\phi,\vec{\lambda}}^{\otimes k}$ is (twisted) $\mathcal{D}$-module.

$$\begin{array}{ccc} & \mathcal{M}^{par}_{G,\vec{\lambda}} = \mathcal{X} \longleftarrow \mathcal{L}_{\phi,\vec{\lambda}} \\ & \downarrow \pi \\ \mathcal{C} \longrightarrow & S \end{array} \qquad S \xrightarrow{\{p_i\}_{i=1}^n} \mathcal{C}$$

Theorem (Biswas-M-Wentworth:21(i))

Let $D = \sqcup p_i(S)$ and $\Phi : R^1\pi_*\mathcal{T}_{\mathcal{C}/S}(-D) \to R^1\pi_*\mathcal{T}_{\mathcal{M}^{par}_{G,\lambda}/S}$. Then,

1. $\cup[\mathcal{L}_{\phi,\vec{\lambda}}] \circ \rho_{Hit} + \Phi = 0$ *(Hitchin:87 for $n = 0$) where ρ_{Hit} is the degree two part of the Hitchin map.*
2. Φ *is an isomorphism. (Narasimhan-Ramanan:75 for $n = 0$).*
3. $\mu_{\mathcal{L}_\phi^{\otimes k}}$ *is an isomorphism and is given by $\cup(k[\mathcal{L}_{\phi,\vec{\lambda}}] - \frac{1}{2}[K_{\mathcal{X}/S}])$.*

Projective operator via symbols: Hitchin-van Geemen-de Jong

Goal: $KS_{\mathcal{X}/S} + \mu_{\mathcal{L}_{\phi,\vec{\lambda}}^{\otimes k}} \circ \rho = 0$. (rewrite (1) using (3)) for some ρ between $\mathcal{T}_S$ and $\pi_* \operatorname{Sym}^2 \mathcal{T}_{\mathcal{X}/S}$.

Candidate symbol:(Biswas-M-Wentworth:21(i)) Consider $\rho_{par} : \mathcal{T}_S \to \pi_* \operatorname{Sym}^2 \mathcal{T}_{\mathcal{X}/S}$ defined by the formula

$$\left(\frac{1}{m_\phi k} + \mu^{-1}_{\mathcal{L}_{\phi,\vec{\lambda}}^{\otimes k}} \circ \left(\cup \frac{1}{2m_\phi k}[K_{\mathcal{M}^{par}_{G,\vec{\lambda}}/S}]\right)\right) \circ \rho_{Hit} \circ KS_{\mathcal{C}/S},$$

where ρ_{Hit} is the degree two part of the Hitchin map and $\mu_{\mathcal{L}_{\phi,\vec{\lambda}}^{\otimes k}}$ is given by $\cup(k[\mathcal{L}_{\phi,\vec{\lambda}}] - \frac{1}{2}[K_{\mathcal{X}/S}])$.

Remark: For non-parabolic, $[K_{\mathcal{M}_{\mathrm{SL}_r}/S}] = 2r[\mathcal{L}]$ and hence $\mu_{\mathcal{L}^{\otimes k}}$ is linear in k and the symbol is just $\frac{1}{r+k}\rho_{Hit} \circ KS_{\mathcal{C}/S}$ as in Hitchin:87.

Parabolic Hitchin connection

- Theorem: (M-Wentworth:21) The candidate symbol defines a flat projective connection for moduli of twisted Spin bundles $\mathcal{M}^{-}_{\mathrm{Spin}(r)}(C)$. The generic Hitchin fibers are dual as abelian varieties to that $\mathcal{M}^{-}_{\mathrm{PSO}(r)}(C)$.
- Theorem: (Biswas-M-Wentworth:21(i)) The candidate symbol ρ_{par} defines a flat projective connection in the parabolic cases.

Remark: Assuming a theta structure Schneiost-Schottenloher:95, Andersen-Bjerre, Pauly-Zakaria. ($n = 0$: Welters:83. Hitchin:87, Axelrod-Della Pietra-Witten:91, Faltings:93, Ginzburg:94, Ramadas:98, Sun-Tsai:04, Ran:06, Baier-Bolognesi-Martens-Pauly:20).

We have extensively used works (Seshadri, Balaji-Biswas-Nagaraj, Balaji-Seshadri:15) on (Γ, G)-bundles, Higgs moduli (Hitchin:87, Nitsure:91, Donagi-Pantev:08) and works of Beilinson-Schechtman:88, Konstevich:87.

- Theorem: M-Wentworth
  A Hitchin type connection exists for moduli spaces of twisted Spin bundles exhibiting a special case for the moduli of parahoric Higgs bundles. Moreover there is a natural duality between the spectral data of the Hitchin maps for Spin(r) and $PSO(r)$.
- Theorem: Biswas-M-Wentworth:21(ii)
  The parabolic Hitchin connection equals the TUY connection on conformal blocks under the natural identification(Laszlo: 98 for non-parabolic).
- Corollary:(BMW:21(ii))
  We get a geometric construction of the KZ equations

$$\nabla_{\frac{\partial}{\partial z_i}} := \frac{\partial}{\partial z_i} - \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{i,j}}{z_i - z_j}$$

  over the bundles of invariants $(V^*_{\lambda_1} \otimes \cdots \otimes V^*_{\lambda_n})^{\mathfrak{g}}$ on the configuration space of points in $\mathbb{C}$.

Gauss-Manin nature of KZ connection

Building on earlier works of Schechtman-Varchenko:94, Ramadas:09, Looijenga:10, Belkale:11, Belkale-M:14.

Theorem (Belkale-Brosnan-M:19)

For each n-tuple $\vec{z}$ of distinct points in $\mathbb{C}$, there is an flat identification of the invariants $(V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_n}^*)^{\mathfrak{g}}$ with

$$(H^M(X_{\vec{z}}, D(\mathbb{L}(-\kappa, \vec{\lambda})) \to H^M(X_{\vec{z}}, (\mathbb{L}(\kappa, \vec{\lambda})))^{Sign},$$

where $X_{\vec{z}}$'s are smooth varieties and $\mathbb{L}$'s build out of the data of weights and κ.

Conformal embedding for adjoint representation

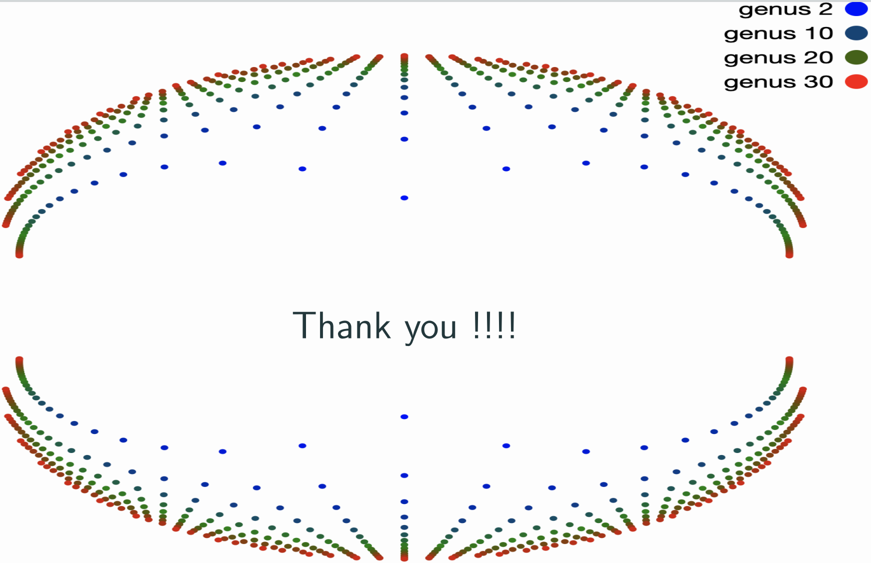
The rank-level duality results discussed above arise out of a special class of embeddings called the conformal embeddings $\mathfrak{s} \hookrightarrow \mathfrak{g}$. Here we consider a special case of such that embedding given as $\mathfrak{g} \hookrightarrow \mathfrak{so}(\mathfrak{g})$.

Theorem (Biswas-M:23)

For any smooth curve C, consider the natural map between the moduli stacks of bundles $\mathcal{M}_G(C) \to \mathcal{M}_{SL(\dim G)}(C)$ induced by the adjoint representation. Then the image of $M_G(C)$ donot lie entirely in the theta divisor given by a choice of a theta characteristic.

As a corollary we generalize a result of Biswas-Hurtubise and Biswas-Hurtubise-Roubtsov about natural isomorphisms between two natural torsors for the cotangent bundle $\Omega^1_{M_G^{rs}}$ on the moduli space of regularly stable G-bundles

Golyshev's canonical strip hypothesis: Belmans-Galkin-M



Thank you !!!!