APPLICATIONS OF THE LIOUVILLE SYMPLECTIC FORM ON THE COTANGENT BUNDLE OF A LOOP GROUP

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Abstract. Let G be a semisimple affine algebraic group defined over $\mathbb{C}$. Consider the Liouville symplectic structure on the total space $T^*G((t))$ of the loop group $G((t))$, where t is a formal parameter. We show that the Liouville symplectic structure on $T^*G((t))$ induces the symplectic structures on the moduli stack of framed Higgs G-bundles on a compact connected Riemann surface X and also on the moduli spaces of framed G-connections on X. These symplectic structures on the on the moduli stack of framed Higgs G-bundles and framed connections on X were constructed earlier. Our results show that they have a common origin. Similar results for these moduli stacks with finite order framings are also obtained.

Contents

1. Introduction

Let X be a compact connected Riemann surface, or equivalently, an irreducible smooth projective curve defined over $\mathbb{C}$. It's canonical line bundle will be denoted by K_X. Fix an effective divisor $\mathbb{D}$ on X. Let G be a semisimple, simply connected, affine algebraic group defined over

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$\mathbb{C}$. Take a principal G-bundle E_G on the curve X, which is same as a holomorphic principal G-bundle on the Riemann surface X. A Higgs field on X is a section

$$\theta \in H^0(X,\, \mathrm{ad}(E_G)\otimes K_X \otimes \mathcal{O}_X(\mathbb{D})),$$

where $\mathrm{ad}(E_G) \longrightarrow X$ is the adjoint bundle of E_G. A Higgs bundle is a principal G-bundle equipped with a Higgs field. A Higgs bundle $(E_G,\, \theta)$ is called semistable (respectively, stable) if for every pair $(P,\, \chi)$, where $P \subsetneq G$ is a parabolic subgroup and χ is a strictly anti-dominant character of P with respect to some Borel subgroup of G contained in P (this means that the line bundle on G/P associated to χ is ample), and for every reduction of structure group $E_P \subset E_G$ such that

$$\theta \in H^0(X,\, \mathrm{ad}(E_P)\otimes K_X \otimes \mathcal{O}_X(\mathbb{D})) \subset H^0(X,\, \mathrm{ad}(E_G)\otimes K_X \otimes \mathcal{O}_X(\mathbb{D})),$$

the inequality

$$\mathrm{degree}(E_P(\chi)) \geq 0 \quad (\text{respectively, } \mathrm{degree}(E_P(\chi)) > 0)$$

holds, where $E_P(\chi) \longrightarrow X$ is the line bundle associated to the principal P-bundle E_P for the character χ of P.

One can construct the moduli space $M_{Higgs}(G)$ of stable Higgs bundles parametrizing the above data. When $\mathbb{D} = 0$, the moduli space $M_{Higgs}(G)$ has a natural symplectic structure [Hi2]. In the general case of $\mathbb{D}$, the moduli space $M_{Higgs}(G)$ has a Poisson structure [Bot], [Ma].

A framed Higgs bundle is a Higgs bundle $(E_G,\, \theta)$ as above equipped with an enhancement given by a trivialization of E_G over the divisor $\mathbb{D}$. A framed Higgs bundle is called stable (respectively, semistable) if the underlying Higgs bundle is stable (respectively, semistable). Let $\widetilde{M}_{Higgs}(G)$ denote the moduli space of framed Higgs bundles on X. This moduli space $\widetilde{M}_{Higgs}(G)$ has a natural symplectic structure [BLP1], [BLP2]. The natural projection $\widetilde{M}_{Higgs}(G) \longrightarrow M_{Higgs}(G)$ is a Poisson map. This result actually extends to the more general context of parabolic Higgs bundles [BLPS].

Let $M_{Conn}(G)$ denote the moduli space of stable principal G-bundles E_G on X equipped with a meromorphic $\mathcal{D}$ connection whose pole is contained in the divisor $\mathbb{D}$ with an appropriate stability condition as for the case of $M_{Higgs}(G)$. Similarly as in the case of meromorphic Higgs bundles, when $\mathbb{D} = 0$, this moduli space has a natural symplectic structure [AB], [Go]. For a general $\mathbb{D}$, the moduli space $M_{Conn}(G)$ has a Poisson structure [Boa1], [Boa2].

One can enhance the structure of meromorphic connections using the notion of framed connections as in the case for frame Higgs bundles. A framed connection is a pair $(E_G,\, \mathcal{D}) \in M_{Conn}(G)$ together with a trivialization of E_G over the divisor $\mathbb{D}$. The moduli space of framed connections $\widetilde{M}_{Conn}(G)$ has a natural symplectic structure [BIKS1], [BIKS2]. Moreover, the natural projection to $M_{Conn}(G)$ from the moduli space of framed connections is a Poisson map.

The nonabelian Hodge correspondence identifies the moduli space Higgs bundles with the moduli space of principal bundles with algebraic connections (same as holomorphic connections) [Si], [Hi1], [Do], [Co]. Similar results hold for the moduli spaces of framed connection. However, this identification is not algebraic or holomorphic but C^∞. Hence symplectic structure on one side of the non-abelian Hodge correspondence do not automatically give rise to symplectic structures on the other side.

The main goal of this paper is to show that all the above symplectic structures have a single common origin. They all originate from the Liouville symplectic structure on the total space $T^*G((t))$ of the loop group $LG := G((t))$, where t is a formal parameter. Here we can think of the loop group as $\mathbb{C}$-valued points of a stack parametrizing G bundles on a curve with a given trivialization on a formal disk D_p as well as on the punctured curve $X\backslash p$. We can also think of $G((t))$ as a Fréchet Lie group.

Here for the rest of the paper and for the simplicity of the exposition, we will assume that the divisor $\mathbb{D} = p$, where p is a point on X. The case of a general effective divisor $\mathbb{D}$ follows directly using the same methods.

Fix a point $p \in X$. Let $L_X G := G(X\backslash\{p\})$ be the space of all algebraic maps from $X\backslash\{p\}$ to the group G. Then the double quotient

$$G[[t]]\backslash LG/L_X G$$

is identified with the space of $\mathbb{C}$-valued points of the moduli stack of principal G-bundles over X. Note that this double quotient is in the category of stacks. On the other hand, $LG/L_X G$ is identified with the space of $\mathbb{C}$-valued points of the moduli stack of principal G-bundles E_G over X equipped with a trivialization of E_G over the formal completion D_p of X along p.

Let $\mathcal{M}_{Higgs}(G)$ denote the moduli stack of principal Higgs G-bundles $(E_G,\, \theta)$ on X equipped with a trivialization of E_G over the formal completion D_p; the Higgs field is allowed to have a pole at p of arbitrary order. The right-translation of action of $L_X G$ on LG produces an action of $L_X G$ on the total space T^*LG of the cotangent bundle of LG. The above moduli stack $\mathcal{M}_{Higgs}(G)$ is a quotient, by the action of $L_X(G)$, of an $L_X(G)$-invariant subbundle $\mathcal{W}$ of T^*LG.

We prove the following (see Theorem 4.2):

Theorem 1.1. *Restrict the Liouville symplectic form on T^*LG to $\mathcal{W}$. This restriction descends to a 2-form on the quotient space $\mathcal{M}_{Higgs}(G) = \mathcal{W}/L_X G$. The 2-form on $\mathcal{M}_{Higgs}(G)$ obtained this way is actually a symplectic form.*

Similar we consider the following version of moduli of connections. Let $\mathcal{M}_{Conn}(G)$ denote the moduli stack of pairs of principal G-bundles (E_G, ∇) on X and a meromorphic connection ∇ such that the principal bundle E_G is equipped with a trivialization of E_G over the formal completion D_p; the connection ∇ is allowed to have a pole at fixed point p on X of arbitrary order. As before, the right-translation of action of $L_X G$ on LG produces an action of $L_X G$ on the total space T^*LG of the cotangent bundle of LG. The above moduli stack $\mathcal{M}_{Conn}(G)$ is a quotient, by the action of $L_X(G)$, of an $L_X(G)$-invariant subbundle $\mathcal{U}$ of T^*LG.

We prove the following (see Theorem 4.2):

Theorem 1.2. *Restrict the Liouville symplectic form on T^*LG to $\mathcal{U}$. This restriction descends to a 2-form on the quotient space $\mathcal{M}_{Conn}(G) = \mathcal{U}/L_X G$. The 2-form on $\mathcal{M}_{Conn}(G)$ obtained this way is actually a symplectic form.*

A key ingredient in the proof of Theorem 1.1 and Theorem 1.2 is Theorem 2.1.

To describe Theorem 2.1, fix finitely many distinct points $\{Q_1,\, \cdots,\, Q_n\} \subset X$. Fix a formal parameter ξ_i at each Q_i, $1 \leq i \leq n$. Take a principal G-bundle E_G on X. Its adjoint bundle will be denoted by $\mathrm{ad}(E_G)$. For each $1 \leq i \leq n$, fix a trivialization of E_G on the formal completion $\widehat{Q}_i$ of X along Q_i, which, in turn, gives a trivialization of $\mathrm{ad}(E_G)$ on $\widehat{Q}_i$. Using these trivializations, we have

$$H^0(X,\, \mathrm{ad}(E_G)(*\sum_{i=1}^n Q_i)) \,:=\, \lim_{j\to\infty} H^0(X,\, V\otimes \mathcal{O}_X(j\sum_{i=1}^n Q_i)) \,\hookrightarrow\, \bigoplus_{i=1}^n \mathfrak{g}\otimes\mathbb{C}((\xi_i)),$$

$$H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(*\sum_{i=1}^n Q_i)) \,:=\, \lim_{j\to\infty} H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(j\sum_{i=1}^n Q_i)) \,\hookrightarrow\, \bigoplus_{i=1}^n \mathfrak{g}\otimes\mathbb{C}((\xi_i))d\xi_i,$$

where $\mathfrak{g}$ is the Lie algebra of G. There is a natural nondegenerate pairing

$$\mathcal{R} \,:\, (\bigoplus_{i=1}^n \mathfrak{g}\otimes\mathbb{C}((\xi_i)))\otimes(\bigoplus_{i=1}^n \mathfrak{g}\otimes\mathbb{C}((\xi_i))d\xi_i) \,\longrightarrow\, \mathbb{C}$$

defined by

$$((X_i \otimes f_i(\xi_i)_{i=1})^n),\, ((Y_i \otimes g_i(\xi_i)d\xi_i)_{i=1}^n) \longmapsto \sum_{i=1}^n (X_i, Y_i)\,\mathrm{Res}_{\xi_i=0}(f_i(\xi_i)g_i(\xi_i)d\xi_i)$$

We refer the reader to equation (2.3)).

Theorem 2.1, which generalizes [Ue, Theorem 1.22], says the following:

Theorem 1.3. *The subspace*

$$H^0(X,\, \mathrm{ad}(E_G)(*\sum_{i=1}^n Q_i)) \ \subset\ \bigoplus_{i=1}^n \mathfrak{g}\otimes \mathbb{C}((\xi_i))$$

and the subspace

$$H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(*\sum_{i=1}^n Q_i)) \ \subset\ \bigoplus_{i=1}^n \mathfrak{g}\otimes \mathbb{C}((\xi_i))d\xi_i$$

are the annihilators of each other under the residue pairing $\mathcal{R}$.

Theorem 1.1 also holds when then order of the pole at p of the Higgs field in bounded by a positive integer m and also the order of the infinitesimal neighborhood of p on which E_G is trivialized is $m-1$; see Theorem 4.5.

Similar results are proved for the moduli stacks of framed connections; see Theorem 5.6 and Theorem 5.7. The proof of these theorems follow along the same line as the proof of Theorem 4.2 and Theorem 5.6 as these stacks can be expressed as a sub-quotient of the cotangent bundle of the Lie group of G.

2. Annihilators and the residue pairing

In this section, we prove some general results about annihilators of adjoint bundle-valued forms on a smooth complex curve under the residue pairing.

Let X be an irreducible smooth complex projective curve or, equivalently, a compact connected Riemann surface. The genus of X will be denoted by g. The canonical line bundle of the curve X will be denoted by K_X.

Let G be a affine algebraic semisimple group defined over $\mathbb{C}$; the Lie algebra of G will be denoted by $\mathfrak{g}$. Let E_G be a principal G-bundle on the curve X. The adjoint vector bundle for E_G, which is the vector bundle on X associated to the principal G-bundle E_G for the adjoint action of G on its Lie algebra $\mathfrak{g}$, will be denoted by $\mathrm{ad}(E_G)$.

Fix finitely many distinct point $Q_1, \cdots, Q_n$ on X. For notational convenience, for a vector bundle V on X and any integer m, the vector bundle

$$V\otimes \mathcal{O}_X(m\sum_{i=1}^n Q_i) \,=\, V\otimes (\mathcal{O}_X(\sum_{i=1}^n Q_i)^{\otimes m}) \longrightarrow X$$

will be denoted by $V(m\sum_{i=1}^n Q_i)$. The direct limit

$$\lim_{j\to\infty} H^0(X,\, V\mathcal{O}_X(j\sum_{i=1}^n Q_i)),$$

constructed using the natural inclusion maps

$$H^0(X,\, V\mathcal{O}_X(j\sum_{i=1}^n Q_i)) \hookrightarrow H^0(X,\, V\mathcal{O}_X((j+m)\sum_{i=1}^n Q_i)),$$

for $m \geq 0$, will be denoted by $H^0(X,\, V\mathcal{O}_X(*\sum_{i=1}^n Q_i))$.

For each $1 \leq i \leq n$, let ξ_i be a choice of formal parameter at the point Q_i. For each $1 \leq i \leq n$, fix a trivialization of E_G on the formal completion $\widehat{Q}_i$ of X along Q_i. So the restriction of $\text{ad}(E_G)$ to $\widehat{Q}_i$ is identified with the trivial Lie algebra bundle over $\widehat{Q}_i$ with fiber $\mathfrak{g}$. Using these trivializations of $\text{ad}(E_G)\big|_{\widehat{Q}_i}$, the Laurent expansions via the formal parameters give us the following inclusion maps:

$$\bigoplus_{i=1}^n \iota_i \ : \quad H^0(X,\, \text{ad}(E_G)(*\textstyle\sum_{i=1}^n Q_i)) \ \hookrightarrow \ \bigoplus_{i=1}^n \mathfrak{g}\otimes \mathbb{C}((\xi_i)), \tag{2.1}$$

$$\bigoplus_{i=1}^n \eta_i \ : \quad H^0(X,\, \text{ad}(E_G)\otimes K_X(*\textstyle\sum_{i=1}^n Q_i)) \ \hookrightarrow \ \bigoplus_{i=1}^n \mathfrak{g}\otimes \mathbb{C}((\xi_i))d\xi_i. \tag{2.2}$$

Let $(-\,,\,-)$ denote the normalized Cartan-Killing form on the semisimple Lie algebra $\mathfrak{g}$. This form induces the following residue pairing

$$\mathcal{R} \ : \ (\bigoplus_{i=1}^n \mathfrak{g}\otimes \mathbb{C}((\xi_i)))\otimes (\bigoplus_{i=1}^n \mathfrak{g}\otimes \mathbb{C}((\xi_i))d\xi_i) \ \longrightarrow \ \mathbb{C} \tag{2.3}$$

$$((X_i\otimes f_i(\xi_i)_{i=1})^n),\ ((Y_i\otimes g_i(\xi_i)d\xi_i)_{i=1}^n) \ \longmapsto \ \sum_{i=1}^n (X_i, Y_i)\,\text{Res}_{\xi_i=0}(f_i(\xi_i)g_i(\xi_i)d\xi_i).$$

Note that $\text{Res}_{\xi_i=0}(f_i(\xi_i)g_i(\xi_i)d\xi_i)$ is well-defined because the coefficients of ξ_i^k in the expansions of $f_i(\xi_i)$ and $g_i(\xi_i)$ vanish for all sufficiently negative k. Moreover the pairing $\mathcal{R}$ is non-degenerate.

The following theorem is a generalization of Theorem 1.22 in [Ue].

Theorem 2.1. *The subspace*

$$H^0(X,\, \text{ad}(E_G)(*\sum_{i=1}^n Q_i))$$

in (2.1) *and the subspace*

$$H^0(X,\, \text{ad}(E_G)\otimes K_X(*\sum_{i=1}^n Q_i))$$

in (2.2) *are the annihilators of each other under the residue pairing* $\mathcal{R}$ *in* (2.3).

Proof. For any pair of positive integers m and N, consider the following short exact sequence of coherent sheaves on X:

$$0 \longrightarrow \text{ad}(E_G)(-m\sum_{i=1}^n Q_i) \longrightarrow \text{ad}(E_G)(N\sum_{i=1}^n Q_i) \longrightarrow \bigoplus_{i=1}^n \bigoplus_{k=-N}^{m-1} \mathfrak{g}\otimes \mathbb{C}\cdot \xi_i^k \longrightarrow 0; \tag{2.4}$$

we have used the chosen trivializations of $\text{ad}(E_G)$ over the formal completions $\widehat{Q}_i$, $1 \leq i \leq n$, to identify the quotient sheaf in (2.4) with $\bigoplus_{i=1}^n \bigoplus_{k=-N}^{m-1} \mathfrak{g}\otimes \mathbb{C}\xi_i^k$. It can be shown that for N sufficiently large,

$$H^1(X,\, \text{ad}(E_G)(N\sum_{i=1}^n Q_i)) \ = \ 0. \tag{2.5}$$

For example, if we take any N such that $Nn + \mu_{\min}(\text{ad}(E_G)) > 2(g-1)$, where $\mu_{\min}(\text{ad}(E_G))$ is the smallest one among the slopes of the successive quotients for the Harder–Narasimhan filtration of the vector bundle $\text{ad}(E_G)$, and $g = \text{genus}(X)$, then using Serre duality we have

$$H^1(X,\, \text{ad}(E_G)(N\sum_{i=1}^n Q_i)) \ = \ H^0(X,\, \text{ad}(E_G)^*(-N\sum_{i=1}^n Q_i)\otimes K_X)^* \ = \ 0,$$

because the given condition that $Nn + \mu_{\min}(\mathrm{ad}(E_G)) > 2(g-1)$ implies that we have

$$\mu_{\max}(\mathrm{ad}(E_G)^*(-N\sum_{i=1}^n Q_i) \otimes K_X) \;=\; \mu_{\max}(\mathrm{ad}(E_G)^*) - Nn - N + 2(g-1)$$

$$=\; -\mu_{\min}(\mathrm{ad}(E_G)) - Nn - N + 2(g-1) \;<\; 0,$$

where $\mu_{\max}(\mathrm{ad}(E_G)^*)$ is the largest one among the slopes of the successive quotients for the Harder–Narasimhan filtration of the dual vector bundle $\mathrm{ad}(E_G)^*$. Here we are using the observation that any locally free coherent sheaf on X, whose $\mu_{\max}$ is negative, does not admit any nonzero section; note that this observation follows immediately from the fact that $\mu(\mathcal{O}_X) = 0$. (See [HL] for the construction and the properties of the Harder–Narasimhan filtration.) Hence (2.5) holds.

We note that using the Cartan-Killing form on $\mathfrak{g}$, we have

$$\mathrm{ad}(E_G) \;=\; \mathrm{ad}(E_G)^*. \tag{2.6}$$

This implies that $\mu_{\max}(\mathrm{ad}(E_G)) = -\mu_{\min}(\mathrm{ad}(E_G))$.

Next we observe that $H^0(X,\, \mathrm{ad}(E_G)(-m\sum_{i=1}^n Q_i)) = 0$ for all m sufficiently large. Indeed, if m is such that $\mu_{\max}(\mathrm{ad}(E_G)) < mn$, then we have $H^0(X,\, \mathrm{ad}(E_G)(-m\sum_{i=1}^n Q_i)) = 0$ because

$$\mu_{\max}(\mathrm{ad}(E_G)(-m\sum_{i=1}^n Q_i)) \;=\; \mu_{\max}(\mathrm{ad}(E_G)) - mn \;<\; 0.$$

So take N and m to be sufficiently large such that

$$H^1(X,\, \mathrm{ad}(E_G)(N\sum_{i=1}^n Q_i)) \;=\; 0 \;=\; H^0(X,\, \mathrm{ad}(E_G)(-m\sum_{i=1}^n Q_i)). \tag{2.7}$$

Consider the following long exact sequence of cohomologies corresponding to the short exact sequence of sheaves in (2.4):

$$0 \longrightarrow H^0(X,\, \mathrm{ad}(E_G)(-m\textstyle\sum_{i=1}^n Q_i)) \longrightarrow H^0(X,\, \mathrm{ad}(E_G)(N\sum_{i=1}^n Q_i)) \xrightarrow{\gamma} \bigoplus_{i=1}^n \bigoplus_{k=-N}^{m-1} \mathfrak{g}\otimes\mathbb{C}\cdot\xi_i^k \xrightarrow{\delta} H^1(X,\, \mathrm{ad}(E_G)(-m\sum_{i=1}^n Q_i)) \longrightarrow H^1(X,\, \mathrm{ad}(E_G)(N\sum_{i=1}^n Q_i)) \longrightarrow 0. \tag{2.8}$$

Using (2.7), this reduces to the following short exact sequence:

$$\longrightarrow H^0(X,\, \mathrm{ad}(E_G)(N\textstyle\sum_{i=1}^n Q_i)) \xrightarrow{\gamma} \bigoplus_{i=1}^n \bigoplus_{k=-N}^{m-1} \mathfrak{g}\otimes\mathbb{C}\xi_i^k \xrightarrow{\delta} H^1(X,\, \mathrm{ad}(E_G)(-m\sum_{i=1} Q_i)) \longrightarrow 0. \tag{2.9}$$

Next we observe that Serre duality, combined with the isomorphism given in (2.6), produces a perfect pairing

$$B \,:\, H^1(X,\, \mathrm{ad}(E_G)(-m\sum_{i=1}^n Q_i)) \otimes H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(m\sum_{i=1}^n Q_i)) \longrightarrow \mathbb{C}. \tag{2.10}$$

So this pairing B identifies the cohomology $H^1(X,\, \mathrm{ad}(E_G)(-m\sum_{i=1}^n Q_i))$ with the dual of $H^0(X, \mathrm{ad}(E_G)\otimes K_X(m\sum_{i=1}^n Q_i))$.

Take $g(\xi_i) \in \bigoplus_{k=-N}^{m-1} \mathfrak{g}\otimes\mathbb{C}\xi_i^k$ for every $1 \leq i \leq n$. Consider

$$\delta((g(x_i)_{i=1}^n) \in H^1(X,\, \mathrm{ad}(E_G)(-m\sum_{i=1} Q_i)),$$

where δ is the homomorphism in (2.9). One can show that

$$B(\delta((g(x_i))_{i=1}^n),\, \tau) \;=\; \mathcal{R}((g(x_i))_{i=1}^n,\, (\eta_i(\tau)_{i=1}^n)) \tag{2.11}$$

for all $\tau \in H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(m\sum_{i=1}^n Q_i))$, where η_i are the homomorphisms in (2.2) and $\mathcal{R}$ is the pairing in (2.3). Indeed, (2.11) follows immediately by comparing the constructions of B and $\mathcal{R}$.

Since B in (2.10) is a perfect pairing, it follows immediately that we have $\delta((g(x_i))_{i=1}^n) = 0$ if and only if

$$B(\delta((g(x_i))_{i=1}^n),\, \tau) \;=\; 0$$

for all $\tau \in H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(m\sum_{i=1}^n Q_i))$. In view of (2.11), from this we conclude that $\delta((g(x_i))_{i=1}^n) = 0$ if and only

$$\mathcal{R}((g(x_i))_{i=1}^n,\, ((\eta_i(\tau)_{i=1}^n)) \;=\; 0$$

for every $\tau \in H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(m\sum_{i=1}^n Q_i))$.

Now by the exactness of equation (2.9), we have $\ker\delta = \mathrm{im}\,\gamma$. Therefore, we have proved the following lemma:

Lemma 2.2. *An element* $\omega \in \bigotimes_{j=1}^n\bigoplus_{k=-N}^{m-1}\mathfrak{g}\otimes\mathbb{C}\xi_j^k$ *lies in the subspace*

$$\gamma(H^0(X,\, \mathrm{ad}(E_G)(N\sum_{j=1}^n Q_j))) \subset \bigoplus_{j=1}^n\bigoplus_{k=-N}^{m-1}\mathfrak{g}\otimes\mathbb{C}\xi_j^k$$

(see (2.9) for γ) if and only if

$$\mathcal{R}(\omega,\, \tau) \;=\; 0$$

for every $\tau \in H^0(C,\, \mathrm{ad}(E_G)\otimes K_X(m\sum_{j=1}^n Q_j))$*, where $\mathcal{R}$ is the residue pairing in (2.3); here* $H^0(C,\, \mathrm{ad}(E_G)\otimes K_X(m\sum_{j=1}^n Q_j))$ *is considered as a subspace of* $\bigoplus_{i=1}^n\mathfrak{g}\otimes\mathbb{C}((\xi_i))d\xi_i$ *using the maps* $(\eta_i)_{i=1}^n$ *in (2.2).*

Lemma 2.2 will be used in completing the proof of Theorem 2.1.

Suppose that an element

$$\boldsymbol{\alpha} \;=\; (\boldsymbol{\alpha}_1,\, \cdots,\, \boldsymbol{\alpha}_n) \in \bigoplus_{i=1}^n \mathfrak{g}\otimes\mathbb{C}((\xi_i))$$

is annihilated by the subspace $H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(*\sum_{i=1}^n Q_i))$ for the pairing $\mathcal{R}$ in (2.3); as before, $H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(*\sum_{i=1}^n Q_i))$ is considered as a subspace of $\bigoplus_{i=1}^n\mathfrak{g}\otimes\mathbb{C}((\xi_i))d\xi_i$ using the maps $(\eta_i)_{i=1}^n$ in (2.2). We can write

$$\boldsymbol{\alpha}_i \;=\; \sum_{k=-N_i}^{\infty} a_k^{(i)}\xi_i^k, \tag{2.12}$$

where $a_k^{(i)} \in \mathfrak{g}$. Fix a positive integer N' sufficiently large such that we have

$$H^1(X,\, \mathrm{ad}(E_G)((N'+j)\sum_{i=1}^n Q_i)) \;=\; 0 \tag{2.13}$$

for all $j \geq 0$; it was observed earlier that it is possible to choose such an integer N' (see (2.5)).

Now define

$$N \ := \ \max\{N_1, \cdots, N_n;\, N'\}, \tag{2.14}$$

where N_i are as in (2.12) and N' is the integer in (2.13).

Recall the above condition on $\boldsymbol{\alpha}$ that it is annihilated by $H^0(X, \operatorname{ad}(E_G) \otimes K_X(* \sum_{i=1}^n Q_i))$ for the pairing $\mathcal{R}$. For any

$$\omega_0 \ \in \ H^0(X, \operatorname{ad}(E_G) \otimes K_X(* \sum_{i=1}^n Q_i)), \tag{2.15}$$

write, using (2.2),

$$\omega_0 \ = \ \bigoplus_{i=1}^n \sum_{j=-M_i}^{+\infty} X_{i,j} \otimes \xi_i^j \cdot d\xi_i, \tag{2.16}$$

where $X_{i,j} \in \mathfrak{g}$. Then we have

$$\mathcal{R}(\boldsymbol{\alpha}, \omega_0) \ = \ \sum_{i=1}^n \sum_{k=-N_i}^{M_i-1} (a_k^{(i)}, X_{i,-(k+1)}). \tag{2.17}$$

For any fixed m big enough, and $1 \le i \le n$, we define a truncation $\boldsymbol{\alpha}_{i,m}$ of $\boldsymbol{\alpha_i}$ as follows:

$$\boldsymbol{\alpha}_{i,m} \ := \ \sum_{k=-N_i}^{m-1} a_k^{(i)} \xi_i^k \tag{2.18}$$

(see (2.12)). Now consider $(\boldsymbol{\alpha}_{i,m})_{i=1}^n$. Note that for

$$m \ > \ \max\{M_1, \cdots, M_n\}, \tag{2.19}$$

the section ω_0 in (2.15) lies in the following subspace

$$\omega_0 \ \in \ H^0(X, \operatorname{ad}(E_G) \otimes K_X(m \sum_{i=1}^n Q_i)) \ \subset \ H^0(X, \operatorname{ad}(E_G) \otimes K_X(* \sum_{i=1}^n Q_i)).$$

Assume that m satisfies the inequality in (2.19). It can be shown that

$$\mathcal{R}(\boldsymbol{\alpha}, \omega_0) \ = \ \mathcal{R}(\boldsymbol{\alpha}_{i,m}, \omega_0). \tag{2.20}$$

To see this, first note that $\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i,m}$ has a zero at each Q_i, $1 \le i \le n$, of order at least m. From the inequality in (2.19) and the expression in (2.16) we conclude that the pairing $(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i,m}, \omega_0) \in \mathbb{C}((\xi_i))$ does note have any pole at Q_i. This immediately implies that (2.20) holds.

Since $\boldsymbol{\alpha}$ is annihilated by $H^0(X, \operatorname{ad}(E_G) \otimes K_X(* \sum_{i=1}^n Q_i))$ for the pairing $\mathcal{R}$, from (2.20) it follows that $(\boldsymbol{\alpha}_{i,m})_{i=1}^n$ is also annihilated by $H^0(X, \operatorname{ad}(E_G) \otimes K_X(m \sum_{i=1}^n Q_i))$ for the pairing $\mathcal{R}$.

As $(\boldsymbol{\alpha}_{i,m})_{i=1}^n$ is annihilated by $H^0(X, \operatorname{ad}(E_G) \otimes K_X(m \sum_{i=1}^n Q_i))$ for the pairing $\mathcal{R}$, we observe that Lemma 2.2 implies that there is a global section

$$\boldsymbol{\alpha}^{(m)} \ \in \ H^0(X, \operatorname{ad}(E_G)(N \sum_{i=1}^n Q_i)) \tag{2.21}$$

(see (2.14) for N) whose Laurent expansion at each Q_i gives $\boldsymbol{\alpha}_{i,m}$. Now further $\boldsymbol{\alpha}^{(m)}$ is also an element of the space $H^0(X, \operatorname{ad}(E_G)(* \sum_{i=1}^n Q_i))$. So for any

$$\widetilde{\omega} \ \in \ H^0(X, \operatorname{ad}(E_G) \otimes K_X(* \sum_{i=1}^n Q_i)),$$

the pairing $(\boldsymbol{\alpha}^{(m)},\, \widetilde{\omega})$ is a meromorphic 1-form on X. Consequently, the total residue of the form $(\boldsymbol{\alpha}^{(m)},\, \widetilde{\omega})$ is zero. Therefore, $\boldsymbol{\alpha}^{(m)}$ in (2.21) is annihilated by any element of $H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(*\sum_{i=1}^n Q_i))$ under the residue pairing.

Hence we conclude that $(\boldsymbol{\alpha}_{i,m})_{i=1}^n$ is annihilated by the entire $H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(*\sum_{i=1}^n Q_i))$ under the pairing $\mathcal{R}$.

Now for each $i \in \{1, \cdots, n\}$, define

$$\boldsymbol{\beta}_i \ := \ \boldsymbol{\alpha}_i - \boldsymbol{\alpha}^{(m)}. \tag{2.22}$$

Observe that for all $i \in \{1, \cdots, n\}$, this $\boldsymbol{\beta}_i$ has a zero at Q_i of order at least m (see (2.12) and (2.18)).

To complete the proof of the theorem, it suffices to show that

$$\boldsymbol{\beta}_i \ = \ 0 \tag{2.23}$$

for every $1 \leq i \leq n$.

Suppose for some k, we have

$$\boldsymbol{\beta}_k \ \neq \ 0. \tag{2.24}$$

Thus we can find $s \geq m$ such that

$$\boldsymbol{\beta}_k \ = \ b_s\xi_k^s + b_{s+1}\xi_k^{s+1} + \ldots, \tag{2.25}$$

with $b_s \neq 0$.

Recall that $\boldsymbol{\alpha}$ and $(\boldsymbol{\alpha}_{i,m})_{i=1}^n$ are both actually annihilated by the entire $H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(*\sum_{i=1}^n Q_i))$. Hence from (2.22) it follows that

$$\mathcal{R}((\boldsymbol{\beta}_i)_{i=1}^n,\, \omega_0) \ = \ 0 \tag{2.26}$$

for all $\omega_0 \in H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(*\sum_{i=1}^n Q_i))$.

Choose m such that

$$m + \mu_{\min}(\mathrm{ad}(E_G)) \ > \ 0.$$

Since $s \geq m$ (see (2.25)), this implies that

$$s + \mu_{\min}(\mathrm{ad}(E_G)) \ > \ 0. \tag{2.27}$$

Consider the following short exact sequence of coherent sheaves on X:

$$0 \longrightarrow \mathrm{ad}(E_G)\otimes K_X\otimes \mathcal{O}_X(sQ_k) \longrightarrow \mathrm{ad}(E_G)\otimes K_X\otimes \mathcal{O}_X((s+1)Q_k) \longrightarrow \mathrm{ad}(E_G)_{Q_k}\otimes (K_X\otimes \mathcal{O}_X((s+1)Q_k))_{Q_k} \longrightarrow 0,$$

where k is as in (2.25) and $\mathrm{ad}(E_G)_{Q_k}\otimes(K_X\otimes\mathcal{O}_X((s+1)Q_k))$ is the fiber of $\mathrm{ad}(E_G)_{Q_k}\otimes(K_X\otimes \mathcal{O}_X((s+1)Q_k))$ over the point Q_k. It gives an exact sequence of cohomologies

$$\longrightarrow H^0(X, \mathrm{ad}(E_G)\otimes K_X\otimes \mathcal{O}_X((s+1)Q_k)) \xrightarrow{\ \rho\ } \mathrm{ad}(E_G)\otimes K_X\otimes \mathcal{O}_X((s+1)Q_k)_{Q_k} \xrightarrow{\ \delta\ } H^1(X, \mathrm{ad}(E_G)_{Q_k}\otimes (K_X\otimes \mathcal{O}_X((s+1)Q_k))) \longrightarrow . \tag{2.28}$$

By Serre duality,

$$H^1(X,\, \mathrm{ad}(E_G)\otimes K_X\otimes \mathcal{O}_X(sQ_k)) \ = \ H^0(X,\, \mathrm{ad}(E_G)^*\otimes \mathcal{O}_X(-sQ_k))^*.$$

We have
$$\mu_{\max}(\mathrm{ad}(E_G)^* \otimes \mathcal{O}_X(-sQ_k)) \;=\; -s - \mu_{\min}(\mathrm{ad}(E_G)) \;<\; 0$$
using (2.27). This implies that we have $H^0(X,\, \mathrm{ad}(E_G)^* \otimes \mathcal{O}_X(-sQ_k)) \;=\; 0$, and hence it follows that
$$H^1(X,\, \mathrm{ad}(E_G) \otimes K_X \otimes \mathcal{O}_X(sQ_k)) \;=\; 0.$$
Consequently, the homomorphism ρ in (2.28) is surjective.

Since ρ in (2.28) is surjective, there is a section
$$\omega' \;\in\; H^0(X,\, \mathrm{ad}(E_G) \otimes K_X \otimes \mathcal{O}_X((s+1)Q_k)) \;\subset\; H^0(X,\, \mathrm{ad}(E_G) \otimes K_X((s+1)\sum_{i=1}^{n} Q_i))$$
with the following property: Let ω'_{s+1} be the coefficient of $\xi_k^{-(s+1)}$ in the Laurent expansion of ω' around the point Q_k (see (2.24) for k); then
$$(b_s,\, \omega'_{s+1}) \;\neq\; 0, \tag{2.29}$$
where b_s is as in (2.25) (recall that $b_s \neq 0$).

Since ω', as a meromorphic section of $\mathrm{ad}(E_G) \otimes K_X$, has pole only at Q_k, and $(\boldsymbol{\beta}_i)_{i=1}^n$ has no pole at any Q_i, we conclude that
$$\mathcal{R}((\boldsymbol{\beta}_i)_{i=1}^n,\, \omega') \;=\; (b_s,\, \omega'_{s+1}) \;\neq\; 0$$
(see (2.29)). But this contradicts (2.26). Therefore, we conclude that (2.23) holds for every $1 \,\leq\, i \,\leq\, n$. This completes the proof of the theorem. □

3. Symplectic Structures

3.1. **Induced symplectic form.** Let V_1 be a complex vector space, not necessarily finite dimensional. Let
$$A \,:\, V_1 \otimes V_1 \,\longrightarrow\, \mathbb{C}$$
be an alternating bilinear form. The form gives an element
$$\omega \,\in\, \bigwedge^2 V_1^*.$$
This element ω of $\bigwedge^2 V_1^*$ induces a linear map
$$\widetilde{\omega} \,:\, V_1 \,\longrightarrow\, V_1^*.$$
Note that there are two possible choices of ω': any $v \,\in\, V_1$ is sent to the map $w \,\longmapsto\, A(v,\, w)$ or to the map $w \,\longmapsto\, A(w,\, v)$. These two homomorphisms differ only by a sign. We say that the form A is *symplectic* if the homomorphism $\widetilde{\omega}$ is injective, in which case $\widetilde{\omega}$ is also called symplectic.

Let A be a symplectic structure on V_1. Let V_2 be a linear subspace of V_1. We have the following sequence of linear maps:
$$0 \,\longrightarrow\, V_2 \,\longrightarrow\, V_1 \,\xrightarrow{\widetilde{\omega}}\, V_1^* \,\longrightarrow\, V_2^* \,\longrightarrow\, 0. \tag{3.1}$$
Let
$$f \,:\, V_2 \,\longrightarrow\, V_2^*$$
be the composition maps in (3.1); let
$$K \,:=\, \mathrm{Ker}(f) \,\subset\, V_2 \tag{3.2}$$
be the kernel of f. Hence f induces an injective homomorphism
$$\phi \,:\, V_2/K \,\hookrightarrow\, V_2^*. \tag{3.3}$$

We address the question which asks whether ϕ induces a symplectic form on V_2/K.

Consider the short exact sequence

$$0 \longrightarrow (V_2/K)^* \xrightarrow{\alpha} V_2^* \xrightarrow{\alpha'} K^* \longrightarrow 0, \tag{3.4}$$

where α is the dual of the quotient map $V \longrightarrow V_2/K$ and the surjective homomorphism $V_2^* \longrightarrow K^*$ is the dual of the inclusion map in (3.2). It is straightforward to see that we have an induced symplectic structure on V_2/K if

$$\mathrm{Im}(\phi) \subseteq \mathrm{Im}(\alpha), \tag{3.5}$$

where ϕ and α are the homomorphisms in (3.3) and (3.4) respectively. Indeed, if (3.5) holds, then ϕ factors through a homomorphism

$$\widetilde{\phi} : V_2/K \hookrightarrow (V_2/K)^*.$$

In other words, $\widetilde{\phi}$ is uniquely determined by the following condition:

$$\alpha \circ \widetilde{\phi} = \phi.$$

This homomorphism $\widetilde{\phi}$ is anti-symmetric because $\widetilde{\omega}$ in (3.1) is so.

To prove that (3.5) holds, take any $v \in V_2$; its image in V_2/K will be denoted by $\widehat{v}$. We have

$$\phi(\widehat{v})(w) = A(v, w)$$

for all $w \in V_2$. Restrict $\phi(\widehat{v})$ to K. For $w \in K$, we have

$$\phi(\widehat{v})(w) = A(v, w) = -A(w, v) = -f(w)(v) = 0$$

(see (3.2)). This implies that (3.5) holds.

Therefore, we have proved the following:

Proposition 3.1. *Let A be a symplectic form on a vector space V_1. Let V_2 be a subspace, and let $K = \mathrm{Ker}(f)$ be the kernel of the homomorphism f (constructed as in (3.2)). Then A induces a symplectic structure $\overline{A}$ on the quotient vector space V_2/K.*

Note that we can describe the above subspace K as

$${}^{L}K = \{v \in V_2 \mid A(v, w) = 0 \ \forall \ w \in V_2\}.$$

Since the form A is bilinear and anti-symmetric, ${}^{L}K$ is a vector subspace of V_2 and it coincides with the subspace

$${}^{R}K = \{v \in V_2 \mid A(w, v) = 0 \ \forall \ w \in V_2\}.$$

3.2. **Canonical symplectic structures.** Let M be a smooth manifold. Consider the cotangent bundle

$$p : T^*M \longrightarrow M. \tag{3.6}$$

We recall that there is a natural 1-form θ on T^*M which is constructed as follows: For any $x \in M$ and any $w \in (T_xM)^*$, we have $\theta(v) = w(dp(v))$ for all $v \in T_w(T^*M)$, where $dp : T(T^*M) \longrightarrow TM$ is the differential of the projection p in (3.6). This form θ is known as the *Liouville one-form.* Then

$$\omega_{T^*M} = d\theta \tag{3.7}$$

is a symplectic form on the total space of T^*M, i.e., it is a non-degenerate closed 2-form on T^*M.

If M is a complex manifold, then θ and ω_{T^*M} are holomorphic forms. If M is a smooth variety, then both θ and ω_{T^*M} are algebraic forms.

3.3. **Liouville form on groups.** Let be $\mathscr{G}$ be a Fréchet Lie group not-necessarily finite dimensional. Then the cotangent bundle $T^*\mathscr{G}$ is trivial and is given by $\mathscr{G} \times \mathfrak{s}^*$, where $\mathfrak{s}$ is the Lie algebra of $\mathscr{G}$. The tangent bundle of $T^*\mathscr{G}$ is just $(\mathscr{G} \times \mathfrak{s}^*) \times (\mathfrak{s} \times \mathfrak{s}^*)$.

Let $p : T^*\mathscr{G} \to \mathscr{G}$ be the projection map. Now consider the derivative of the projection map

$$dp : T(T^*(\mathscr{G})) \to T\mathscr{G}.$$

If $(\alpha, \beta) \in \mathfrak{s} \times \mathfrak{s}^*$, the Liouville one-form is defined by the $\theta(\alpha, \beta) = \beta(\alpha)$. Similarly the symplectic form is defined by the formula

$$\omega((v, \phi), (w, \psi)) = \phi(w) - \psi(v).$$

3.4. **Symplectic structures on the quotient.** Let M be smooth manifold equipped with a nondegenerate symmetric bilinear form ω which is nondegenerate. Here non-degeneracy means the following: Let

$$\omega' \, : \, TM \, \hookrightarrow \, T^*M \tag{3.8}$$

be the homomorphism produced by this nondegenerate symmetric bilinear form ω; it should be clarified that non-degeneracy of ω means that the homomorphism ω' is fiberwise injective. Note that this condition implies that ω' is an isomorphism if M is a finite dimensional manifold.

Assume that a Lie group $\mathscr{G}$ acts on M such that the above nondegenerate symmetric bilinear form ω on M is preserved by the action of $\mathscr{G}$. The Lie algebra of $\mathscr{G}$ will be denoted by $\mathfrak{g}_0$. The action of $\mathscr{G}$ on M produces a homomorphism

$$\phi \, : \, M \times \mathfrak{g}_0 \, \longrightarrow \, TM \tag{3.9}$$

from the trivial vector bundle $M \times \mathfrak{g}_0 \longrightarrow M$ on M with fiber $\mathfrak{g}_0$.

The action of $\mathscr{G}$ on M induces an action of $\mathscr{G}$ on the tangent bundle TM, and as well as on the cotangent bundle T^*M. Now consider the subsheaf $\mathcal{F}^{\mathscr{G}}$ of TM given by the $\mathscr{G}$ orbits; its fiber at a point $m \in M$ is $T_m(\mathscr{G} \cdot m)$. More precisely, $\mathcal{F}^{\mathscr{G}}$ is the image of the homomorphism ϕ in (3.9), so we have

$$\mathcal{F}^{\mathscr{G}} \, = \, \text{image}(\phi) \, \subset \, TM. \tag{3.10}$$

The annihilator of $\mathcal{F}^{\mathscr{G}}$ for the nondegenerate symmetric bilinear form ω on M will be denoted by $(\mathcal{F}^{\mathscr{G}})^{\perp}$. (Since ω is symmetric, $\mathcal{F}^{\mathscr{G}}$ does not depend on the two choices available to define the annihilator.)

Let

$$\mathcal{V} \, := \, \omega'((\mathcal{F}^{\mathscr{G}})^{\perp}) \, \subset \, T^*M \tag{3.11}$$

be the image of the annihilator $(\mathcal{F}^{\mathscr{G}})^{\perp}$ under the homomorphism ω' in (3.8), where $\mathcal{F}^{\mathscr{G}}$ is defined in (3.10). Then $\mathcal{V}$ is a subsheaf of the cotangent bundle T^*M of a manifold M. Note that $\mathcal{V}$ plays the role of the annihilator of $\mathcal{F}^{\mathscr{G}}$ under the pairing ω.

Consider the action of $\mathscr{G}$ on T^*M induced by the action of $\mathscr{G}$ on M. The following lemma shows that the subspace $\mathcal{V}$ in (3.11) is preserved by this action of $\mathscr{G}$.

Lemma 3.2. *The action of $\mathscr{G}$ on T^*M induced by the action of $\mathscr{G}$ on M, preserves $\mathcal{V}$ constructed in* (3.11).

Proof. Consider the action of $\mathscr{G}$ on TM induced by the action of $\mathscr{G}$ on M. The homomorphism ϕ in (3.9) is evidently $\mathscr{G}$–equivariant for the adjoint action of $\mathscr{G}$ on its Lie algebra $\mathfrak{g}_0$ and the above actions of $\mathscr{G}$ on M and TM. This immediately implies that the action of $\mathscr{G}$ on TM preserves the subsheaf $\mathcal{F}^{\mathscr{G}} \subset TM$ in (3.10).

Recall the given condition that the action of $\mathscr{G}$ on M preserves the nondegenerate symmetric bilinear form ω. Since the action of $\mathscr{G}$ on TM preserves $\mathcal{F}^{\mathscr{G}}$, this implies that the action of $\mathscr{G}$ on TM also preserves $(\mathcal{F}^{\mathscr{G}})^{\perp}$.

Consider the action of $\mathscr{G}$ on T^*M induced by the action of $\mathscr{G}$ on M. The homomorphism ω' in (3.8) is $\mathscr{G}$–equivariant because the nondegenerate symmetric bilinear form ω is preserved by the action of $\mathscr{G}$ on M. Since the action of $\mathscr{G}$ preserves $(\mathcal{F}^{\mathscr{G}})^{\perp}$, and ω' is $\mathscr{G}$–equivariant, we conclude that the action of $\mathscr{G}$ on T^*M preserves $\omega'((\mathcal{F}^{\mathscr{G}})^{\perp}) = \mathcal{V}$. □

Consider the action of $\mathscr{G}$ on $\mathcal{V}$ obtained in Lemma 3.2. Since the natural map

$$p : T^*M \longrightarrow M \tag{3.12}$$

is $\mathscr{G}$–equivariant, the natural projection $\mathcal{V} \longrightarrow M$ is also $\mathscr{G}$–equivariant. Consequently, we have

$$\mathcal{V}_{\mathscr{G}} := \mathcal{V}/\mathscr{G} \longrightarrow M/\mathscr{G}. \tag{3.13}$$

Now consider the Liouville one-form θ_M on T^*M defined in Section 3.2. The following lemma says that the action of $\mathscr{G}$ on T^*M preserves θ_M.

Lemma 3.3. *The action of $\mathscr{G}$ on T^*M, induced by the action of $\mathscr{G}$ on M, preserves the Liouville one-form θ_M on T^*M defined in Section 3.2.*

Proof. This a straight-forward computation using the definition of the Liouville one-form θ_M. □

Restrict the Liouville one-form θ_M to $\mathcal{V}$ (defined in (3.11)). Let

$$\theta'_M \in H^0(\mathcal{V}, T^*\mathcal{V}) \tag{3.14}$$

be the restriction of θ_M to $\mathcal{V}$.

Our aim is to find sufficient conditions ensuring the following:

(1) The 1-form θ'_M on $\mathcal{V}$ (see (3.14)) descends to a 1-form ϑ on the quotient $\mathcal{V}_{\mathscr{G}}$ in (3.13).
(2) The exterior derivative $d\vartheta$ on $\mathcal{V}_{\mathscr{G}}$ is non-degenerate.

The following lemma ensures the first one of the above two requirements actually holds.

Lemma 3.4. *The one-form θ'_M on $\mathcal{V}$ (defined in (3.14)) descends to the quotient $\mathcal{V}_{\mathscr{G}}$ in (3.13).*

Proof. As before, $\mathfrak{g}_0$ denotes the Lie algebra of $\mathscr{G}$. The action of $\mathscr{G}$ on T^*M, induced by the action of $\mathscr{G}$ on M, produces a homomorphism

$$\widetilde{\eta} : (T^*M) \times \mathfrak{g}_0 \longrightarrow T(T^*M). \tag{3.15}$$

Consider the projection p in (3.12). Let

$$dp : T(T^*M) \longrightarrow TM \tag{3.16}$$

be the differential of p. Let

$$\eta := (dp) \circ \widetilde{\eta} : (T^*M) \times \mathfrak{g}_0 \longrightarrow TM \tag{3.17}$$

be the composition of maps, where $\widetilde{\eta}$ is constructed in (3.15).

From Lemma 3.2 we know that the action of $\mathscr{G}$ on T^*M preserves $\mathcal{V}$. Also, Lemma 3.3 says that $\mathscr{G}$ preserves θ_M. Consequently, it suffices to prove the following statement:

For any $x \in M$, $u \in \mathcal{V}_x \subset T^*_x M$ and $v \in \mathfrak{g}_0$,

$$u(\eta(u, v)) = 0, \tag{3.18}$$

where η is the map in (3.17). (Note that $\eta(u, v) \in T_x M$ and $u(\eta(u, v)) \in \mathbb{C}$ because $u \in T^*_x M$.)

To prove (3.18), first note that the maps ϕ (in (3.9)) and η (in (3.17)) are related as follows: For any $y \in M$, $u' \in T^*_y M$ and $v' \in \mathfrak{g}_0$,

$$\eta(u', v') = \phi(p(u'), v'), \tag{3.19}$$

where p is the projection in (3.12). Clearly, we have $\phi(p(u'),\, v') \,\in\, \mathcal{F}^{\mathscr{G}}_y$, where $\mathcal{F}^{\mathscr{G}}$ is defined in (3.10). Therefore, for any

$$t \,\in\, ((\mathcal{F}^{\mathscr{G}})^{\perp})_y \,\subset\, T_yM,$$

we have $\omega(y)(t,\, \phi(p(u'),\, v')) \,=\, 0$, where ω is the nondegenerate symmetric bilinear form on M. This implies that

$$\theta_M(\omega'(t))(\phi(p(u'),\, v')) \,=\, \omega'(t)(\phi(p(u'),\, v')) \,=\, 0.$$

So (3.19) gives that $\theta_M(\omega'(t))(\eta(u',\, v')) \,=\, 0$. But $\theta_M(\omega'(t))(\eta(u',\, v')) \,=\, \omega'(t)(\eta(u',\, v'))$, and hence we conclude that $\omega'(t)(\eta(u',\, v')) \,=\, 0$. From this it follows immediately that (3.18) holds. This completes the proof of the lemma. □

Let

$$\vartheta \in H^0(\mathcal{V}_{\mathscr{G}}, T^*\mathcal{V}_{\mathscr{G}}) \tag{3.20}$$

denote the unique 1-form on the quotient $\mathcal{V}_{\mathscr{G}}$ in (3.13) whose pullback to $\mathcal{V}$ is θ'_M; the existence of ϑ is ensured by Lemma 3.4. Consequently, the pullback of the 2-form $d\vartheta$ to $\mathcal{V}$ coincides with $d\theta'_M$.

Proposition 3.5. *The exterior derivative $d\vartheta$ (see (3.20) for ϑ) is a non-degenerate 2-form on $\mathcal{V}_{\mathscr{G}}$. In particular the total space if $\mathcal{V}_{\mathscr{G}} \longrightarrow M/\mathscr{G}$ inherits a symplectic structure from the symplectic structure $d\theta_M$ on T^*M.*

Proof. In view of Proposition 3.1, the non-degeneracy of $d\vartheta$ is in fact a consequence of the non-degeneracy of $d\theta_M$. To see this, recall that $\mathcal{V}$ is preserved by the action of $\mathscr{G}$ on T^*M (see Lemma 3.2). Let

$$\varphi \,:\, \mathcal{V} \times \mathfrak{g}_0 \longrightarrow T\mathcal{V} \tag{3.21}$$

be the homomorphism given by the action of $\mathscr{G}$ on $\mathcal{V}$, where $\mathfrak{g}_0$, as before, is the Lie algebra of $\mathscr{G}$.

Take a point

$$q \,\in\, \omega'((\mathcal{F}^{\mathscr{G}})^{\perp}) \,=\, \mathcal{V}$$

(see (3.11)). The image of q in $\mathcal{V}/\mathscr{G} \,=\, \mathcal{V}_{\mathscr{G}}$ will be denoted by $\overline{q}$. It can be shown that the tangent space $T_{\overline{q}}\mathcal{V}_{\mathscr{G}}$ has a natural identification

$$T_{\overline{q}}\mathcal{V}_{\mathscr{G}} \,=\, (T_q\mathcal{V})/(\varphi(q,\, \mathfrak{g}_0)), \tag{3.22}$$

where φ is the map in (3.21). Indeed, (3.22) is an immediate consequence of the properties of a quotient space.

The Liouville symplectic form $d\theta_M$ on T^*M (see (3.7)) produces a a fiber-wise injective homomorphism

$$\beta \,:\, T(T^*M) \hookrightarrow T^*(T^*M) \tag{3.23}$$

(see (3.8)).

Using the inclusion map $\mathcal{V} \hookrightarrow T^*M$ in (3.11), we have the maps

$$T_q\mathcal{V} \hookrightarrow T_q(T^*M) \xrightarrow{\beta(q)} T^*_q(T^*M) \twoheadrightarrow T^*_q\mathcal{V}, \tag{3.24}$$

where β is the homomorphism in (3.23). Let

$$\rho \,:\, T_q\mathcal{V} \longrightarrow T^*_q\mathcal{V}$$

be the composition of maps in (3.24). It is straightforward to check that kernel(ρ) coincides with $\varphi(q,\, \mathfrak{g}_0)$. From (3.22) we know that the quotient $(T_q\mathcal{V})/(\varphi(q,\, \mathfrak{g}_0))$ coincides with $T_{\overline{q}}\mathcal{V}_{\mathscr{G}}$. Consequently, using Proposition 3.1 we conclude that $d\vartheta$ is nondegenerate. Hence $d\vartheta$ is a symplectic form on $\mathcal{V}_{\mathscr{G}}$. □

4. Symplectic structure on moduli of Higgs bundles with framings

Let G be a semisimple and simply connected affine algebraic group defined over $\mathbb{C}$. The Lie algebra of G will be denoted by $\mathfrak{g}$. Let X be an irreducible smooth complex projective curve equipped with a marked point $p \in X$. We first recall the uniformization of the stack of principal G-bundles on X.

4.1. **Uniformization of principal G-bundles.** Let t be a formal parameter considered as a holomorphic coordinate at the point $p \in X$. By Harder's theorem, any principal G-bundle on $X\backslash\{p\}$ is trivial [Ha]. Hence by the uniformization theorem [Fa, KNR, BL], the following is obtained:

The $\mathbb{C}$-valued points of the moduli stack of principal G-bundles on the projective curve X can be described as the double quotient

$$\mathrm{Bun}_G(X) \;=\; G[[t]]\backslash G((t))/G(X\backslash\{p\}), \tag{4.1}$$

where $G(X\backslash\{p\})$ is the space of algebraic maps from $X\backslash\{p\}$ to the group G. For notational conveniences, the loop group $G((t))$ is also denoted by LG, while the group $G[[t]]$ of positive loops is denoted by L^+G and the subgroup $G(X\backslash\{p\})$ is denoted by L_XG. So, (4.1) can be re-written as

$$\mathrm{Bun}_G(X) \;=\; L^+G\backslash LG/L_XG.$$

We can consider the elements of the loop group LG as the $\mathbb{C}$-valued points of the moduli stack of principal G-bundles on X equipped with a chosen trivialization on the complement $X\backslash\{p\}$ and a chosen trivialization on the formal disc D_p around the point $\{p\}$. Similarly the ind-variety $L^+G\backslash LG$ parametrizes principal G-bundle on X equipped with a chosen trivialization on the complement $X\backslash\{p\}$.

Now consider the cotangent bundle T^*LG of the loop group LG. The cotangent bundle T^*LG is trivial, in fact, it is identified with the trivial vector bundle

$$S \;:=\; LG \times \left(\mathfrak{g} \otimes K_{D_p}((t))\right); \tag{4.2}$$

here we have identified $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ using the normalized Cartan-Killing form $(-,-)$ on $\mathfrak{g}$. The term K_{D_p} in (4.2) is the canonical bundle of a formal neighborhood D_p of $p \in X$. The group LG acts on the cotangent bundle S of LG.

The loop group LG carries a natural nondegenerate symmetric bilinear form. It is defined as follows:

$$\langle A \otimes t^m,\, B \otimes t^n \rangle \;=\; \delta_{n+m,0}\,(A,\, B)\,, \tag{4.3}$$

where $A,\, B \in \mathfrak{g}$, and $(-,-)$ is the normalized Cartan-Killing form on $\mathfrak{g}$, while $\delta_{i,j} = 0$ if $i \neq j$ and it is 1 if $i = j$ (see [PS]). Note that (4.3) defines a bilinear form $\langle -,- \rangle$ on the vector space $\mathfrak{g}((t))$ using bilinearity. The bilinear form on $\mathfrak{g}((t))$ is evidently symmetric. It is also nondegenerate, meaning the homomorphism

$$\mathfrak{g}((t)) \;\longrightarrow\; \mathfrak{g}((t))^* \tag{4.4}$$

given by the pairing is injective. To see the injectivity of the homomorphism in (4.4), for any $A \otimes t^m$, where $A \in \mathfrak{g}$, take any $B \in \mathfrak{g}$ such that $(A,\, B) \neq 0$. Now, clearly we have $\langle A \otimes t^m,\, B \otimes t^{-m} \rangle \neq 0$, and hence the bilinear form on $\mathfrak{g}((t))$ is nondegenerate.

So (4.3) defines a nondegenerate symmetric bilinear form on the tangent space of LG at the identity element of LG. Now extend this to a nondegenerate symmetric bilinear form on LG using left-translations.

This nondegenerate symmetric bilinear form on LG given by (4.3) will be denoted by ω. As in (3.8), let

$$\omega' \,:\, T(LG) \,\hookrightarrow\, T^*(LG) \tag{4.5}$$

be the homomorphism given by ω; note that ω' is fiber-wise injective because

- ω' injective on the fiber $T_e(LG)$ over the identity element of LG, and
- ω' is LG–equivariant.

From the point of view of Proposition 3.5, the role of M in that proposition will be played by LG while the role of $\mathcal{G}$ will be played by L_XG. In the rest of this section we will consider various subbundles of T^*LG to which Proposition 3.5 can be applied.

4.2. **Principal bundles with meromorphic Higgs fields.** Let E_G be a principal G-bundle on X. The adjoint bundle for E_G will be denoted by $\text{ad}(E_G)$; we recall that $\text{ad}(E_G)$ is the Lie algebra bundle on X associated to the principal G-bundle E_G for the adjoint action of G on its Lie algebra $\mathfrak{g}$. A *meromorphic Higgs field* on E_G is a meromorphic section of the direct limit

$$\varphi \,\in\, H^0(X,\, \text{ad}(E_G)\otimes K_X(*p)) \,=\, \lim_{i\to\infty} H^0(X,\, \text{ad}(E_G)\otimes K_X\mathcal{O}_X(ip)), \tag{4.6}$$

where K_X, as before, is the canonical line bundle of X; the above direct limit is constructed using the natural inclusion maps $\text{ad}(E_G)\otimes \mathcal{O}_X(jp) \,\hookrightarrow\, \text{ad}(E_G)\otimes \mathcal{O}_X((j+k)p)$ where $k \,\geq\, 0$.

A *meromorphic Higgs G-bundle* is a principal G-bundle on X equipped with a meromorphic Higgs field.

We will construct a subbundle

$$\mathcal{W} \,\subset\, T^*LG \tag{4.7}$$

of the cotangent bundle T^*LG. To describe the fibers of $\mathcal{W}$ point-wise, take any point $\alpha \,\in\, LG$. The point α gives a principal G-bundle E_G on X with a given trivialization of E_G on $X\backslash\{p\}$ and a given trivialization of E_G on formal completion D_p. Now consider the adjoint vector bundle $\text{ad}(E_G)$ on X. Note that the elements of $\mathfrak{g}\otimes K_{D_p}((t))$ can be considered as sections of $(\text{ad}(E_G)\otimes K_X)_{D_p}$ with pole of arbitrary order at p. The fiber $\mathcal{W}_\alpha$ consists of all elements of $\mathfrak{g}\otimes K_{D_p}((t))$ that extend to a section of $\text{ad}(E_G)\otimes K_X$ over the complement $X\setminus\{p\}$. So the only point of X where such a section can have pole is p; the order, at p, of the pole of this meromorphic section can be arbitrary.

Recall that $L_XG \,\hookrightarrow\, LG$, and consequently L_XG acts on LG via left-translations. This action of L_XG on LG induces actions of L_XG on both TLG and T^*LG.

Lemma 4.1. *The action of L_XG on T^*LG, induced by the action of L_XG on LG, preserve the subbundle $\mathcal{W}$ in* (4.7).

Proof. We recall that an element α of LG gives a principal G-bundle E_G on X with given trivializations of E_G over $X\backslash\{p\}$ and the formal completion D_p. It is easy to see that we have $\alpha \,\in\, L_XG \,\subset\, LG$ if and only if the trivialization of E_G over D_p extends to a trivialization of E_G over entire X. So we get two trivializations of of $E_G\big\vert_{X\backslash\{p\}}$: One given directly by α and the other obtained by extending, to entire X, the trivialization of $E_G\big\vert_{D_p}$ given by α. These two trivializations of $E_G\big\vert_{X\backslash\{p\}}$ differ by an automorphism of $E_G\big\vert_{X\backslash\{p\}}$.

Now take any $\beta \,\in\, LG$. it gives principal G-bundle F_G on X with given trivializations of F_G over $X\backslash\{p\}$ and D_p. As before, take any $\alpha \,\in\, L_XG$. Then the element $\beta\alpha \,\in\, LG$ gives the same principal G-bundle F_G on X (the principal G-bundle given by β), and the trivialization of F_G over D_p for $\beta\alpha$ remains unchanged (it coincides with the one given by β). But the trivialization of F_G over $X\backslash\{p\}$ for $\beta\alpha$ changes by the automorphism of the trivial principal G-bundle given by α.

Take $\beta \in LG$ as above. Then the fiber $\mathcal{W}_\beta$ of $\mathcal{W}$ (see (4.7)) over β is canonically identified with the space of all meromorphic Higgs fields on the principal G-bundle F_G given by β.

From the above descriptions of $\mathcal{W}_\beta$, and the action of L_XG on LG, it follows immediately that the action of L_XG on T^*LG, induced by the action of L_XG on LG, preserve the subbundle $\mathcal{V}$ in (4.7). ☐

As mentioned before, from the point of view of Proposition 3.5 the role of $\mathscr{G}$ will be played by L_XG. Consider the quotient $\mathcal{W}/L_XG$ which is a subbundle of the quotient $(T^*LG)\,/L_XG$. Note that

$$\mathcal{W}/L_XG \ \subset\ (T^*LG)\,/L_XG$$

are vector bundles on the space LG/L_XG. It may be mentioned that $\mathcal{W}/L_XG$ is precisely the moduli stack of principal G-bundles E_G on X, equipped with

- an arbitrary order framing of E_G at p (meaning a trivialization of E_G over D_p), and
- a meromorphic Higgs field on E_G.

We are in a position to prove the following theorem.

Theorem 4.2. *Consider the moduli stack $\mathcal{W}/L_XG$ parametrizing principal G-bundles on X equipped with an arbitrary order framing at p and a meromorphic Higgs field. It inherits a canonical symplectic structure constructed using the Liouville symplectic structure on T^*LG.*

Proof. We need to put ourselves in the set-up of Proposition 3.5 in order to apply it. As in (3.10), construct the subsheaf

$$\mathcal{F}^{L_XG} \ \subset\ T(LG) \tag{4.8}$$

using the left-translation action of the subgroup L_XG on LG. Let

$$(\mathcal{F}^{L_XG})^{\perp} \subset T(LG)$$

be the annihilator of $\mathcal{F}^{L_XG}$ in (4.8) for the nondegenerate symmetric bilinear form ω on LG (see (4.5) and (4.3)). So we have

$$\mathcal{V} \ :=\ \omega'(\mathcal{F}^{L_XG}) \ \subset\ T^*(LG), \tag{4.9}$$

where ω' is the homomorphism in (4.5).

To prove the theorem it is enough to show that the vector subbundle $\mathcal{V} \subset T^*LG$ (see (4.9)) coincides with the subbundle $\mathcal{W} \ \subset\ T^*LG$ in (4.7).

As before, t is a formal parameter considered as a holomorphic coordinate at the point $p \in X$. Take any element $\alpha \in LG$. The fiber $T_\alpha(LG)$ of the tangent bundle $T(LG)$ over the point α is identified with $\mathfrak{g}((t))$. Indeed, this follows immediately from the fact that the Lie algebra of the loop group LG is $\mathfrak{g}((t))$.

Let E_G denote the principal G-bundle on X given by $\alpha \in LG$. Recall that α gives a trivialization of the principal G-bundle E_G over the formal disc D_p around the point $p \in X$. This trivialization of the principal G-bundle E_G over D_p produces a trivialization of the adjoint vector bundle $\text{ad}(E_G)$ over D_p. More precisely, the restriction of $\text{ad}(E_G)$ to D_p is identified with the trivial Lie algebra bundle $D_p \times \mathfrak{g} \longrightarrow D_p$ over D_p with fiber $\mathfrak{g}$.

Using this trivialization of $\text{ad}(E_G)$ over D_p, for any $\sigma \in H^0(X,\, \text{ad}(E_G)\otimes \mathcal{O}_X(jp))$, where $j \geq 1$, by taking the Laurent expansion of σ around p we get an element of $\mathfrak{g}((t)) = T_\alpha(LG)$. Therefore, we have an injective homomorphism

$$H^0(X,\, \text{ad}(E_G)\otimes \mathcal{O}_X(*p)) \ :=\ \lim_{j\to\infty} H^0(X,\, \text{ad}(E_G)\otimes \mathcal{O}_X(jp)) \hookrightarrow \mathfrak{g}((t)) \ =\ T_\alpha(LG). \tag{4.10}$$

It should be mentioned that the above homomorphism $H^0(X,\, \text{ad}(E_G)\otimes \mathcal{O}_X(*p)) \hookrightarrow \mathfrak{g}((t))$ is Lie algebra structure preserving.

The fiber $(\mathcal{F}^{L_XG})_\alpha$ of $\mathcal{F}^{L_XG}$ (see (4.8)) is the Lie algebra $H^0(X,\, \mathrm{ad}(E_G)\otimes\mathcal{O}_X(*p))$; it should be clarified that $H^0(X,\, \mathrm{ad}(E_G)\otimes\mathcal{O}_X(*p))$ is considered as a subspace of $T_\alpha(LG) = \mathfrak{g}((t))$ using the homomorphism in (4.10). That $(\mathcal{F}^{L_XG})_\alpha = H^0(X,\, \mathrm{ad}(E_G)\otimes\mathcal{O}_X(*p))$ is a straightforward consequence of the fact that the Lie algebra of L_XG is

$$\mathfrak{g}\otimes H^0(X,\mathcal{O}_X(*p)) \;=\; \mathfrak{g}\otimes(\lim_{j\to\infty} H^0(X,\,\mathcal{O}_X(jp))).$$

Now from Theorem 2.1 it follows immediately that

$$(H^0(X,\mathrm{ad}(E_G)(*p)))^\perp \;=\; (H^0(X,\mathrm{ad}(E_G)\otimes K_X(*p))).$$

This proves the assertion that the vector subbundle $\mathcal{V}\subset T^*LG$ (see (4.9)) actually coincides with the subbundle $\mathcal{W}\subset T^*LG$ in (4.7). As observed before, this statement completes the proof of the theorem. □

4.3. **Higgs fields with pole of bounded order.** Fix a positive integer k. We will consider Higgs fields with pole at p of order at most k. Let E_G be a principal G-bundle on X. A *Higgs field on E_G with a pole of order k* is an element of $H^0(X,\, \mathrm{ad}(E_G)\otimes K_X\otimes\mathcal{O}_X(kp))$, where $\mathrm{ad}(E_G)$ is the adjoint bundle for E_G.

Before we proceed further we recall the notion of framing of a principal G bundle at a point $p\in X$.

Definition 4.3. A *k-th order framing* of a principal G bundle E_G at a point p is a choice of a trivialization of $E_{G|(k+1)p}$.

We now consider the moduli stack of principal G-bundles on X with a k-th order framing and a Higgs field with pole of order k. Following the strategy of Section 4.2, a symplectic structure on it will be constructed.

Consider $\mathcal{W}$ constructed in (4.7). Let $\mathcal{W}^k$ be the subbundle of $\mathcal{W}$ which is described fiber-wise as follows: Take any $\alpha\in LG$. Denote by E_G the principal G-bundle on X given by α. Recall that the fiber $\mathcal{W}_\alpha$ of $\mathcal{W}$ over α consists of all elements of $\mathfrak{g}\otimes K_{D_p}((t))$ that extend to a meromorphic section of $\mathrm{ad}(E_G)\otimes K_X$ with allowed pole only at the fixed point p (so they are holomorphic on the complement $X\setminus\{p\}$). Similarly define $\mathcal{W}^k_\alpha$ be the subspace of $\mathcal{W}_\alpha$ whose elements have pole of order at most k at p.

Let $G^k[[t]]$ be the subgroup of the group of positive loops $G[[t]]$ consisting of elements of the form $e+\sum_{j=0}^\infty g_jt^{k+j}$, where e is the identity element of G. Note that the group $G^k[[t]]\times L_XG$ acts on the loop group LG and hence it also acts on T^*LG.

We have the following analog of Lemma 4.1.

Lemma 4.4. *The action of the group $G^k[[t]]\times L_XG$ on T^*LG preserves the above subbundle $\mathcal{W}^k$, thus producing a vector bundle $\mathcal{W}^k_{G^k[[t]]\times L_XG}$ on $G^k[[t]]\backslash LG/L_XG$.*

Proof. The Lie algebra of $G^k[[t]]$ is the subspace

$$t^k\cdot\mathfrak{g}[[t]] \;\subset\; \mathfrak{g}[[t]].$$

Consider the residue pairing $\mathcal{R}$ in (2.3). It can be shown that the annihilator of the above subspace $t^k\cdot\mathfrak{g}[[t]]$ for $\mathcal{R}$ is $t^{-k}\mathfrak{g}[[t]]dt$. Indeed, clearly,

$$\mathcal{R}(t^k\cdot\mathfrak{g}[[t]],\, t^{-k}\mathfrak{g}[[t]]dt) \;=\; 0$$

because for any $v\in t^k\cdot\mathfrak{g}[[t]]$ and $w\in t^{-k}\mathfrak{g}[[t]]dt$, their tensor product $v\otimes w$ does not have a pole. So the annihilator of $t^k\cdot\mathfrak{g}[[t]]$ contains $t^{-k}\mathfrak{g}[[t]]dt$.

To prove that the annihilator of $t^k \cdot \mathfrak{g}[[t]]$ is contained in $t^{-k}\mathfrak{g}[[t]]dt$, take any

$$w \in \mathfrak{g}((t))dt \setminus t^{-k}\mathfrak{g}[[t]]dt$$

lying in the complement. Let $t^{-k-\ell}\beta$ be the first nonzero term of w; so $\ell \geq 1$ and $\beta \in \mathfrak{g}$. Take any $\beta' \in \mathfrak{g}$ such that

$$(\beta\,,\,\beta') \neq 0,$$

where $(-\,,\,-)$ is the normalized Cartan-Killing form on $\mathfrak{g}$. Now note that

$$\mathcal{R}(t^{k+\ell-1}\beta',\, w) = (\beta\,,\,\beta') \neq 0.$$

Consequently, the annihilator of $t^k \cdot \mathfrak{g}[[t]]$ is contained in $t^{-k}\mathfrak{g}[[t]]dt$.

In view of the above observation, the lemma follows by using the argument in the proof of Lemma 4.1. $\square$

Lemma 4.4 allows us to apply Proposition 3.5, and we get the following theorem.

Theorem 4.5. *Consider the moduli stack parametrizing the principal G-bundles E_G on X with k-th order framing of E_G at p and a meromorphic Higgs field on E_G with a pole of order at most k at p. This moduli stack has a canonical symplectic structure coming from the Liouville symplectic structure on T^*LG.*

The proof is similar to the proof of Theorem 4.2 and we omit the details.

5. Principal bundles with framings and meromorphic connections

5.1. **Meromorphic connections.** As before, X is an irreducible smooth complex projective curve. Fix a marked point $p \in X$. Let $\varpi : E_G \longrightarrow X$ be a principal G-bundle on X. The Atiyah bundle $\text{At}(E_G)$ of E_G is the quotient $(TE_G)/G \longrightarrow E_G/G = X$, which a vector bundle on X. Let

$$0 \longrightarrow \text{ad}(E_G) \longrightarrow \text{At}(E_G) \xrightarrow{d\varpi} TX \longrightarrow 0 \tag{5.1}$$

be the Atiyah exact sequence on X, where $d\varpi$ is the differential of the above projection ϖ (see [At]); note that $\text{ad}(E_G) = T_\varpi/G$, where $T_\varpi \subset TE_G$ is the relative tangent bundle for the projection ϖ.

We recall from [At] that an algebraic connection on E_G is a homomorphism

$$\mathcal{D} : TX \longrightarrow \text{At}(E_G)$$

such that $(d\varpi)\circ\mathcal{D} = \text{Id}_{TX}$, where $d\varpi$ is the homomorphism in (5.1). A *meromorphic connection* on E_G is a homomorphism

$$\mathcal{D} : (TX)\otimes\mathcal{O}_X(-np) \longrightarrow \text{At}(E_G),$$

where n is some nonnegative integer, such that $(d\varpi)\circ\mathcal{D} = \text{Id}_{(TX)\otimes\mathcal{O}_X(-np)}$; note that we have

$$(TX)\otimes\mathcal{O}_X(-np) \subset TX$$

because it is assumed that $n \geq 0$, so the composition $(d\varpi)\circ\mathcal{D}$ makes sense.

If $\mathcal{D} : (TX)\otimes\mathcal{O}_X(-np) \longrightarrow \text{At}(E_G)$ is a homomorphism such that

$$(d\varpi)\circ\mathcal{D} = \text{Id}_{(TX)\otimes\mathcal{O}_X(-np)},$$

then $\mathcal{D}$ will be a called a meromorphic connection on E_G with a pole of order at most k.

For the trivial principal G-bundle $E^0_G = M \times G$ on any smooth complex variety M, we have $\text{At}(E^0_G) = \text{ad}(E^0_G) \oplus TM$. The algebraic connection on E^0_G defined by the natural inclusion map

$$TM \hookrightarrow \text{ad}(E_G) \oplus TM = \text{At}(E^0_G)$$

is called the trivial connection on E^0_G.

Every principal G-bundle F_G on X admits a meromorphic connection. Indeed, this follows immediately from the fact that the restriction of F_G to the complement $X \setminus \{p\}$ is trivial; the trivial connection on $F_G\big|_{X\setminus\{p\}}$ is a meromorphic connection on F_G. The space of all meromorphic connections on F_G is an affine space for the vector space $H^0(X,\, \mathrm{ad}(F_G)\otimes K_X\otimes \mathcal{O}_X(*p))$. Recall that $H^0(X,\, \mathrm{ad}(F_G)\otimes K_X\otimes \mathcal{O}_X(*p))$ is the space of all meromorphic Higgs fields on the principal G-bundle F_G.

Take any element $\alpha \in LG$. Recall that α gives a principal G-bundle E_G on X, and

- a trivialization of E_G over the complement $X \setminus \{p\}$, and
- a trivialization of E_G over the formal disc D_p around the point $p \in X$.

Consider the trivial connection on $E_G\big|_{X\setminus\{p\}}$ given by the above trivialization of E_G over $X\setminus\{p\}$. It evidently defines a meromorphic connection on E_G. So for each $\alpha \in LG$, the corresponding principal G-bundle E_G on X is equipped with a meromorphic connection given by α. This meromorphic connection on the principal G-bundle E_G on X given by α will be denoted by

$$\mathcal{D}_\alpha. \tag{5.2}$$

Lemma 5.1. *Take any $\alpha \in LG$, and let E_G be the principal G-bundle on X corresponding to α. Then the space of all meromorphic connections on E_G is canonically identified with the space of all meromorphic Higgs fields on E_G.*

Proof. Take any meromorphic Higgs field

$$\varphi \in H^0(X,\, \mathrm{ad}(E_G)\otimes K_X(*p)) = \lim_{i\to\infty} H^0(X,\, \mathrm{ad}(E_G)\otimes K_X\mathcal{O}_X(ip))$$

on E_G (see (4.6)). Consider $\mathcal{D}_\alpha + \varphi$, where $\mathcal{D}_\alpha$ is the meromorphic connection on E_G given by α (see (5.2)). It is evident that $\mathcal{D}_\alpha + \varphi$ is a meromorphic connection on the principal G-bundle E_G. Conversely, if $\mathcal{D}$ is a meromorphic connection on the principal G-bundle E_G, then $\mathcal{D} - \mathcal{D}_\alpha$ is a meromorphic Higgs field on E_G. ☐

From Lemma 5.1 it follows immediately that for any element $\alpha \in LG$, the fiber $\mathcal{W}_\alpha$ of $\mathcal{W}$ (see (4.7)) over the point α is identified with the space of all meromorphic connections on the principal G-bundle E_G given by α.

Let $\mathcal{U}$ denote the space of all pairs of the form $(\alpha,\, \mathcal{D})$, where $\alpha \in LG$ and $\mathcal{D}$ is a meromorphic connection on the principal G-bundle E_G on X given by α. We have the natural projection

$$\Phi : \mathcal{U} \longrightarrow LG \tag{5.3}$$

that sends any $(\alpha,\, \mathcal{D}) \in \mathcal{U}$ to α. So the fiber $\mathcal{U}_\alpha = \Phi^{-1}(\alpha)$, where $\alpha \in LG$, is the space of meromorphic connection on the principal G-bundle on X given by α.

Lemma 5.1 has the following corollary:

Corollary 5.2. *The fiber bundle $\mathcal{U} \longrightarrow LG$ in (5.3) is canonically identified with the fiber bundle $\mathcal{W}$ in (4.7). In particular, $\mathcal{U}$ is sub-vector bundle of the cotangent bundle T^*LG (because $\mathcal{W}$ is so).*

We recall that the action of L_XG on T^*LG, induced by the action of L_XG on LG, preserves the subbundle $\mathcal{W}$ (see Lemma 4.1). Using the identification of $\mathcal{U}$ with $\mathcal{W}$ given by Corollary 5.2, the action of L_XG on $\mathcal{W}$ produces an action of L_XG on $\mathcal{U}$. Clearly, the map Φ in (5.3) is equivariant for the actions of L_XG on $\mathcal{U}$ and LG. However, $\mathcal{U}$ has a *different* action of L_XG which also satisfies the condition that the map Φ is equivariant. This action of L_XG on $\mathcal{U}$ is actually constructed using a different action of L_XG on T^*LG. We will now describe the new action of L_XG on T^*LG.

Take any $(\alpha,\, \theta) \in LG \times (K_{D_p} \otimes \mathfrak{g}((\xi)))$ and $g \in L_XG$. Then define the action

$$(\alpha,\, \theta)\cdot g \;=\; (\alpha g,\, g^{-1}\mathcal{D}_\alpha(g) + \mathrm{Ad}(g)(\theta)), \tag{5.4}$$

where $\mathcal{D}_\alpha$ is the meromorphic connection in (5.2). It is straight-forward to check that (5.4) defines an action of L_XG on T^*LG. The natural projection $T^*LG \longrightarrow LG$ remains L_XG-equivariant for this new action of L_XG.

Lemma 5.3. *The action of L_XG on T^*LG in (5.4) preserves the subbundle $\mathcal{W} \subset T^*LG$ in in (4.7).*

Proof. Let E_G denote the principal G-bundle on X given by α. In (5.4), assume that θ extends to a section of $\mathrm{ad}(E_G)\otimes K_X$ over the entire complement $X\setminus\{p\}$. Since $g \in L_XG$, this implies that $g^{-1}\mathcal{D}_\alpha(g)$ is defined on entire $X\setminus\{p\}$; recall that $\mathcal{D}_\alpha$ in (5.2) is a regular connection on $E_G\big|_{X\setminus\{p\}} \longrightarrow X\setminus\{p\}$. Also, $\mathrm{Ad}(g)(\theta)$ in (5.4) is evidently defined over entire $X\setminus\{p\}$, because both g and θ are defined over $X\setminus\{p\}$. From these it follows immediately that the action of L_XG on T^*LG in (5.4) preserves the subbundle $\mathcal{W}$. □

From Lemma 5.3 we conclude that the action of L_XG on T^*LG in (5.4) induces an action on the subbundle $\mathcal{W}$ in in (4.7). Therefore, using Corollary 5.2, we have the following:

Corollary 5.4. *Consider the action of L_XG on $\mathcal{W}$ induced by the action of L_XG on T^*LG in (5.4). Using the identification of $\mathcal{W}$ with $\mathcal{U}$ in Corollary 5.2, this action of L_XG on T^*LG produces an action of L_XG on $\mathcal{U}$.*

Lemma 5.5. *The action of L_XG on T^*LG in (5.4) preserves the Liouville symplectic form on T^*LG.*

Proof. This is a straight-forward computation. It should be clarified that the Liouville 1-form (see Section 3.3) is not preserved by the action of L_XG, but its exterior derivative, namely the Liouville symplectic form on T^*LG, is preserved by the action of L_XG. This follows using the fact that any connection on a principal bundle on a smooth complex curve is automatically integrable. (Any connection on a principal bundle on X is integrable because $\bigwedge^2 \Omega^1_X \,=\, \bigwedge^2 K_X \,=\, 0$.) □

Consider the action of L_XG on $\mathcal{U}$ obtained in Corollary 5.4. The corresponding quotient

$$\mathcal{M} \;:=\; \mathcal{U}/L_XG \tag{5.5}$$

is the moduli stack of principal G-bundles F_G on X equipped with

- a trivialization of $F_G\big|_{D_p} \longrightarrow D_p$ on the formal disc D_p around the point $p \in X$, and
- a meromorphic connection on F_G.

The following theorem is an analog of Theorem 4.2 for the moduli stack $\mathcal{M}$ defined (5.5).

Theorem 5.6. *The moduli stack $\mathcal{M}_{Conn}(G)$ in (5.5), which parametrizes the principal G-bundles on X with a meromorphic connection and a trivialization over D_p, has a canonical symplectic structure coming from Liouville symplectic structure on the total space of the cotangent bundle T^*LG.*

Proof. Just like Theorem 4.2, this theorem will also be proved using Proposition 3.5. To get into the set-up of Proposition 3.5, set $\mathscr{G}$ in Proposition 3.5 to be L_XG. Also, set $M \,=\, LG$ in Proposition 3.5. Set ω in Proposition 3.5 to be the nondegenerate symmetric bilinear form ω on LG constructed in (4.3).

Define $\mathcal{F}^{\mathcal{G}}$ as in (3.10), and then define $\mathcal{V}$ as (3.11). Then $\mathcal{W}$ coincides with $\mathcal{V}$, which follow using Theorem 2.1. Therefore, $\mathcal{U}$ is identified with $\mathcal{V}$, because $\mathcal{U}$ is identified with $\mathcal{W}$. Now the theorem follows from Proposition 3.5. □

5.2. **Singular connections with pole of bounded order.** As in Section 4.3, let k be a positive integer and consider principal G bundles E_G on X which is equipped with

- a framing of order k (see Definition 4.3) at the point $p \in X$, and
- a meromorphic connection with a pole of order at most k at the point p.

Recall the fiber bundle $\Phi : \mathcal{U} \longrightarrow LG$ constructed in (5.3). Let

$$\mathcal{U}^k \subset \mathcal{U}$$

be the subbundle whose fiber over any point $\alpha \in LG$ consists of all principal G-bundle E_G on X equipped with a regular connection on $E_G\big|_{X\setminus\{p\}}$ whose pole at p has order at most k.

As in Section 4.2, the group $G^k[[t]] \times L_X G$ acts on the loop group LG and hence also on T^*LG. Thus applying Proposition 3.5, we get the following:

Theorem 5.7. *Let $\mathcal{M}^k_{Conn}(G)$ denote the moduli stack parametrizing the principal G-bundles on X equipped with a k-th order framing at the point p and a meromorphic connection with pole of order at most k at p. Then $\mathcal{M}^k_{Conn}(G)$ inherits a canonical symplectic structure coming from the symplectic structure on T^*LG.*

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